PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Asymptotics of a class of multidimensional Laplace–type integrals. II. Treble integrals

D. Kaminski and R. B. Paris

Phil. Trans. R. Soc. Lond. A 1998 **356**, 625-667
doi: 10.1098/rsta.1998.0180

The Royal Society

Asymptotics of a class of multidimensional Laplace-type integrals. II. Treble integrals

By D. Kaminski† and R. B. Paris
Division of Mathematical Sciences, University of Abertay Dundee, Dundee DD1 1HG, UK

Received 17 May 1996; accepted 16 January 1997

Contents

		PAGE
1.	Introduction	625
2.	Asymptotics with one internal point	627
3.	Asymptotics with two internal points	630
4.	Asymptotics with three internal points	638
	(*a*) Treatment of the series of integrals I_1	641
	(*b*) Treatment of the series of integrals I_2	647
	(*c*) Treatment of the series of integrals I_3	651
	(*d*) Final form for the asymptotic expansion	653
5.	Linear-algebraic formulation	655
6.	Non-triangular faces	657
7.	Numerical examples	661
8.	Closing remarks	665
	References	667

This paper is an extension to three (and more) dimensions of a technique developed to obtain the algebraic asymptotic behaviour of a class of double Laplace-type integrals. The method proceeds by representing treble Laplace-type integrals as iterated Mellin–Barnes integrals, followed by judicious application of residue theory to determine new asymptotic expansions. The class of 'phase functions' of the Laplace integrals to which the analysis applies is restricted to 'polynomials' (non-integral powers are permitted) with an isolated, though possibly highly degenerate, critical point at the origin. The determination of which residues to use in constructing the expansions is characterized in elementary geometric terms. Numerical examples highlighting the use of the expansions are supplied.

Keywords: asymptotic expansions; iterated Mellin–Barnes integrals; Newton diagram; single critical point

1. Introduction

This paper is a sequel to Kaminski & Paris (1998) (henceforth referenced simply as (I)), which described a technique for the development of asymptotic expansions of

† Permanent address: Department of Mathematics and Computer Science, University of Lethbridge, 4401 University Drive, Lethbridge AB, Canada T1K 3M4.

two-dimensional Laplace-type integrals with polynomial phase (non-integer powers permitted). The technique, a careful sequence of residue theory evaluations of iterated Mellin–Barnes integrals, was carried out in (I) with an eye to exploiting geometric information in the Newton diagram of the phase function. This programme is continued in this paper, but due to the richer geometric structure of the Newton diagram for phases of higher dimension, we concentrate our efforts on so-called 'convex' cases.

A brief survey of strategies employed in developing asymptotic expansions of Laplace-type integrals is provided in (I) and will not be reproduced here. Definitions involving the Newton diagram are also to be found in (I), as are standard order estimates for Γ functions of large argument. We do recall here the method of representing multiple dimension Laplace-type integrals as iterated Mellin–Barnes integrals.

Let us define for each $r = 1, 2, \ldots, k$,

$$\delta_r = 1 - \frac{m_r}{\mu} - \frac{n_r}{\nu} - \frac{p_r}{\eta},$$

and set

$$\boldsymbol{m} = (m_1, m_2, \ldots, m_k), \quad \boldsymbol{n} = (n_1, n_2, \ldots, n_k), \quad \boldsymbol{p} = (p_1, p_2, \ldots, p_k),$$
$$\boldsymbol{t} = (t_1, t_2, \ldots, t_k), \quad \boldsymbol{\delta} = (\delta_1, \delta_2, \ldots, \delta_k).$$

As in (I), we apply the integral representation

$$\mathrm{e}^{-z} = \frac{1}{2\pi\mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \Gamma(t) z^{-t}\, \mathrm{d}t, \quad |\arg z| < \tfrac{1}{2}\pi, \quad z \neq 0,$$

to each factor $\exp(-\lambda c_r x^{m_r} y^{n_r} z^{p_r})$ in the integrand of

$$I(\lambda) = \int_0^\infty \int_0^\infty \int_0^\infty \exp\left[-\lambda \left(x^\mu + y^\nu + z^\eta + \sum_{r=1}^{k} c_r x^{m_r} y^{n_r} z^{p_r} \right) \right] \mathrm{d}x\mathrm{d}y\mathrm{d}z, \tag{1.1}$$

to arrive, after interchanging the order of integration, at the representation

$$I(\lambda) = \frac{\lambda^{-1/\mu-1/\nu-1/\eta}}{\mu\nu\eta} \left(\frac{1}{2\pi\mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \right)^k \Gamma(\boldsymbol{t}) \Gamma\left(\frac{1 - \boldsymbol{m}\cdot\boldsymbol{t}}{\mu} \right) \times \Gamma\left(\frac{1 - \boldsymbol{n}\cdot\boldsymbol{t}}{\nu} \right) \Gamma\left(\frac{1 - \boldsymbol{p}\cdot\boldsymbol{t}}{\eta} \right) \boldsymbol{c}^{-\boldsymbol{t}} \lambda^{-\boldsymbol{\delta}\cdot\boldsymbol{t}}\, \mathrm{d}\boldsymbol{t}, \tag{1.2}$$

where we have set, as in (I),

$$\Gamma(\boldsymbol{t}) = \Gamma(t_1)\Gamma(t_2)\cdots\Gamma(t_k), \quad \boldsymbol{c}^{-\boldsymbol{t}} = c_1^{-t_1} c_2^{-t_2} \cdots c_k^{-t_k} \quad \text{and} \quad \mathrm{d}\boldsymbol{t} = \mathrm{d}t_1 \mathrm{d}t_2 \cdots \mathrm{d}t_k.$$

The 'dot' appearing between two vector quantities is just the usual Euclidean dot product. The integration contours are indented to the right away from the origin to avoid the pole of the integrand present there, and the constants $c_1, \ldots, c_k$, are positive (see, however, (I, § 3) for a more complete account of the range of admissible values of the c_r).

Observe that, as in (I), the number of independent variables $t_1, \ldots, t_k$ in (1.2) is governed by the number of terms in the sum in the integrand of (1.1), but unlike (I), the higher dimensionality of (1.1) is reflected in the integrand of (1.2) by the presence of an additional Γ function, $\Gamma((1 - \boldsymbol{p}\cdot\boldsymbol{t})/\nu)$, corresponding to the additional spatial variable z. Because of this, we can expect the asymptotic expansions we develop to contain asymptotic series with an additional Γ function appearing in the summand of each series.

The representation (1.2) can be readily extended to deal with higher-dimensional Laplace-type integrals. Each additional dimension of such a Laplace integral manifests itself in the Mellin–Barnes integral (1.2) by the appearance of another factor such as $\Gamma((1-\boldsymbol{m}\cdot\boldsymbol{t})/\mu)$, while the dimensionality of (1.2) is governed by the number of terms in the summation present in the phase, as presented in (1.1). The dimensionality of the associated Newton diagram will also increase and with it, the accompanying difficulty of geometric analysis in four or more dimensions. If only the asymptotic behaviour of a high-dimensional Laplace-type integral is required with no geometric interpretation, the methods employed in this paper and (I) can be used with little modification.

In (I) we devoted some attention to the problem of developing expansions when some internal points were not vertices of the Newton diagram. We saw there that these non-vertex internal points did not produce asymptotic series, and manifested their presence only by appearing in the arguments of Γ functions in the summands of asymptotic series associated with faces of the Newton diagram. Accordingly, we shall focus our attention in this paper only on the 'convex' case, where every internal point $P_i=(m_i,n_i,p_i)$ is a vertex of the Newton diagram.

2. Asymptotics with one internal point

By setting $k=1$, we can recast (1.1) and (1.2) as

$$\begin{aligned} I(\lambda) &= \int_0^\infty\int_0^\infty\int_0^\infty \exp[-\lambda(x^\mu+y^\nu+z^\eta+x^{m_1}y^{n_1}z^{p_1})]\,\mathrm{d}x\mathrm{d}y\mathrm{d}z \\ &= \frac{\lambda^{-1/\mu-1/\nu-1/\eta}}{\mu\nu\eta}\frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\Gamma(t)\Gamma\left(\frac{1-m_1t}{\mu}\right)\Gamma\left(\frac{1-n_1t}{\nu}\right)\Gamma\left(\frac{1-p_1t}{\eta}\right)\lambda^{-\delta_1 t}\,\mathrm{d}t, \end{aligned} \tag{2.1}$$

where, for ease of presentation, the value of the constant c_1 in (1.1) has been set to unity. The analysis of $I(\lambda)$ in this case closely parallels that of (I, § 4), and so we restrict our attention only to the 'convex' case in which the point $P_1=(m_1,n_1,p_1)$ lies in front of the back face of the Newton diagram.

The back face of the Newton diagram for our class of 'phases' is simply the plane generated by $(\mu,0,0)$, $(0,\nu,0)$ and $(0,0,\eta)$ and given by

$$\frac{m}{\mu}+\frac{n}{\nu}+\frac{p}{\eta}-1=0. \tag{2.2}$$

If P_1 lies on the same side of the back face as the origin (i.e. in front of the back face), then P_1 must satisfy $m_1/\mu+n_1/\nu+p_1/\eta-1<0$ or equivalently, $\delta_1>0$. Otherwise, the reverse inequalities hold if P_1 lies behind the back face (if P_1 lies on the back face, then (2.2) must be satisfied in which case $\delta_1=0$).

In the convex case, $\delta_1>0$, and the Newton diagram comprises three triangular faces, each formed by the vertex P_1 and any two of $(\mu,0,0)$, $(0,\nu,0)$ and $(0,0,\eta)$ (see figure 1). Together with the back face, these triangular faces generate a tetrahedron. The planes generated by each of the faces of the Newton diagram are easy to obtain.

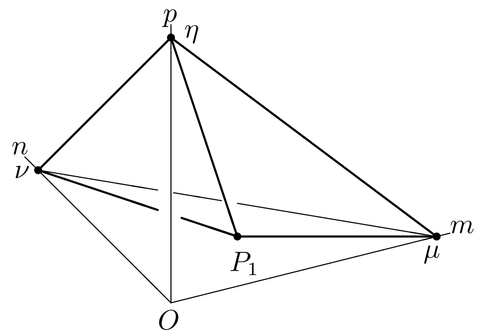

Figure 1. Newton diagram for the convex case with one internal point.

If we denote by $\Pi(Q_1, Q_2, Q_3)$ the plane through the points Q_1, Q_2 and Q_3, then we find that $\Pi_{\mu\nu p_1} \equiv \Pi((\mu, 0, 0), (0, \nu, 0), P_1)$ has equation

$$\frac{m}{\mu} + \frac{n}{\nu} + \frac{\mu\nu - \mu n_1 - m_1\nu}{\mu\nu p_1} p = 1,$$

$\Pi_{\mu n_1 \eta} \equiv \Pi((\mu, 0, 0), P_1, (0, 0, \eta))$ has equation

$$\frac{m}{\mu} + \frac{\mu\eta - m_1\eta - \mu p_1}{\mu n_1 \eta} n + \frac{p}{\eta} = 1$$

and $\Pi_{m_1\nu\eta} \equiv \Pi(P_1, (0, \nu, 0), (0, 0, \eta))$ has equation

$$\frac{\nu\eta - \nu p_1 - n_1\eta}{m_1\nu\eta} m + \frac{n}{\nu} + \frac{p}{\eta} = 1.$$

We shall find it convenient to determine the abscissa m^* of the point of intersection of each of these planes with the diagonal line $m = n = p$. For each plane Π above, let us use the subscript of the plane to identify the particular value of m^* arising from the intersection with this diagonal. Thus, the planes $\Pi_{\mu\nu p_1}$, $\Pi_{\mu n_1\eta}$ and $\Pi_{m_1\nu\eta}$ meet the diagonal at (m^*, m^*, m^*) where, in turn, each m^* satisfies

$$\left.\begin{aligned} -1/m^*_{\mu\nu p_1} &= -1/\mu - 1/\nu - 1/p_1 + n_1/\nu p_1 + m_1/\mu p_1, \\ -1/m^*_{\mu n_1\eta} &= -1/\mu - 1/n_1 - 1/\eta + p_1/n_1\eta + m_1/\mu n_1, \\ -1/m^*_{m_1\nu\eta} &= -1/m_1 - 1/\nu - 1/\eta + p_1/m_1\eta + n_1/m_1\nu. \end{aligned}\right\} \tag{2.3}$$

These quantities will be of value when considering the order of leading terms in the expansion of $I(\lambda)$.

The asymptotics of (2.1) are easily obtained. As in (I, § 4), we find the dominant behaviour of the logarithm of the modulus of the integrand, putting $t = \rho e^{i\theta}$, $|\theta| < \frac{1}{2}\pi$, to be

$$\delta_1 \rho \cos\theta \log\rho + \mathcal{O}(\rho),$$

which, since P_1 is in front of the back face, tends to $+\infty$ as $\rho \to \infty$. Accordingly, the asymptotics of (2.1) can be obtained by displacing the integration contour to the right. As we do so, we find that poles in three sequences must be considered. From poles of the factor $\Gamma((1 - m_1 t)/\mu)$ we obtain the formal series of contributions (omitting a leading factor of $\lambda^{-1/\mu - 1/\nu - 1/\eta}$)

$$I_{m_1\nu\eta} \equiv I_1 = \frac{1}{m_1\nu\eta} \sum_k \frac{(-1)^k}{k!} \Gamma\left(\frac{K}{m_1}\right) \Gamma\left(\frac{m_1 - n_1 K}{m_1\nu}\right) \Gamma\left(\frac{m_1 - p_1 K}{m_1\eta}\right) \lambda^{-\delta_1 K/m_1}, \tag{2.4}$$

while those of $\Gamma((1-n_1t)/\nu)$ and $\Gamma((1-p_1t)/\eta)$ similarly give rise to

$$I_{\mu n_1 \eta} \equiv I_2 = \frac{1}{\mu n_1 \eta} \sum_k \frac{(-1)^k}{k!} \Gamma\left(\frac{n_1 - m_1 K'}{\mu n_1}\right) \Gamma\left(\frac{K'}{n_1}\right) \Gamma\left(\frac{n_1 - p_1 K'}{n_1 \eta}\right) \lambda^{-\delta_1 K'/n_1}, \tag{2.5}$$

$$I_{\mu \nu p_1} \equiv I_3 = \frac{1}{\mu \nu p_1} \sum_k \frac{(-1)^k}{k!} \Gamma\left(\frac{p_1 - m_1 K''}{\mu p_1}\right) \Gamma\left(\frac{p_1 - n_1 K''}{\nu p_1}\right) \Gamma\left(\frac{K''}{p_1}\right) \lambda^{-\delta_1 K''/p_1}, \tag{2.6}$$

respectively, provided none of the three sequences of poles shares a common point, and where we have written

$$K = 1 + \mu k, \quad K' = 1 + \nu k, \quad K'' = 1 + \eta k, \tag{2.7}$$

for notational convenience. Assembling these together, we arrive at the asymptotic expansion

$$I(\lambda) \sim \lambda^{-1/\mu - 1/\nu - 1/\eta}(I_{m_1 \nu \eta} + I_{\mu n_1 \eta} + I_{\mu \nu p_1}), \tag{2.8}$$

as $\lambda \to \infty$. The correspondence we observed in (I) between the number of series constituting the compound asymptotic expansion of $I(\lambda)$ and the number of faces of the Newton diagram continues to hold here, with the series $I_{m_1 \nu \eta}$ corresponding to the face with vertices P_1, $(0, \nu, 0)$ and $(0, 0, \eta)$, the series $I_{\mu n_1 \eta}$ corresponding to the face with vertices P_1, $(\mu, 0, 0)$ and $(0, 0, \eta)$, and finally $I_{\mu \nu p_1}$ corresponding to the face generated by P_1, $(\mu, 0, 0)$ and $(0, \nu, 0)$.

The leading terms in each of the series $I_{m_1 \nu \eta}$, $I_{\mu n_1 \eta}$ and $I_{\mu \nu p_1}$ gives rise to the asymptotic approximation

$$\begin{aligned} I(\lambda) \sim{}& \frac{1}{m_1 \nu \eta} \Gamma\left(\frac{1}{m_1}\right) \Gamma\left(\frac{m_1 - n_1}{m_1 \nu}\right) \Gamma\left(\frac{m_1 - p_1}{m_1 \eta}\right) \lambda^{-1/m^*_{m_1 \nu \eta}} \\ &+ \frac{1}{\mu n_1 \eta} \Gamma\left(\frac{n_1 - m_1}{\mu n_1}\right) \Gamma\left(\frac{1}{n_1}\right) \Gamma\left(\frac{n_1 - p_1}{n_1 \eta}\right) \lambda^{-1/m^*_{\mu n_1 \eta}} \\ &+ \frac{1}{\mu \nu p_1} \Gamma\left(\frac{p_1 - m_1}{\mu p_1}\right) \Gamma\left(\frac{p_1 - n_1}{\nu p_1}\right) \Gamma\left(\frac{1}{p_1}\right) \lambda^{-1/m^*_{\mu \nu p_1}}, \end{aligned}$$

in view of (2.3). The determination of which of these three terms dominates can be resolved by examining the geometry of the Newton diagram. If the diagonal $m = n = p$ punctures a face, then the diagonal must puncture the planes containing the remaining two faces at points along the diagonal between the origin and point of intersection of the diagonal with the Newton diagram. This provides a means of ordering the quantities $-1/m^*_{m_1 \nu \eta}$, $-1/m^*_{\mu n_1 \eta}$ and $-1/m^*_{\mu \nu p_1}$ of (2.3). We remark that the face punctured by the diagonal corresponds to the term in the preceding approximation for which all Γ functions have positive argument.

Additionally, in the case where the diagonal punctures a face, it is possible to have double or treble poles in the higher-order terms. This will happen whenever the arguments of the Γ functions in (2.4)–(2.6) (i.e. expressions such as $(m_1 - n_1 K)/m_1 \nu$, $(n_1 - p_1 K')/n_1 \eta$, $(p_1 - m_1 K'')/\mu p_1$ and so on) is a non-positive integer. Only by choosing special values for μ, ν, η, m_1, n_1 and p_1 can this be avoided.

If the diagonal meets the Newton diagram in an edge, then two of m_1, n_1 and p_1 must be the same, say $m_1 = n_1 > p_1$. In this event, the Γ functions appearing

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

in $I_{m_1\nu\eta}$ and $I_{\mu n_1\eta}$ will have $k = 0$ corresponding to a double pole, so that the leading term in the asymptotic expansion of $I(\lambda)$ will contain a logarithmic factor. Furthermore, if μ/ν is rational, other double poles will appear and if $\mu = \nu$, all terms in the series will have double poles. Higher order terms may even have some treble poles, but these will not be present in the leading behaviour.

Finally, if the diagonal meets the Newton diagram at the vertex P_1, then $m_1 = n_1 = p_1$ and the $k = 0$ terms in each of the expansions will arise from a treble pole. The possibility of double or treble poles in other terms in the expansion depends on the rationality of ratios of pairs of μ, ν and η. The completely symmetrical case of $\mu = \nu = \eta$ produces poles that are all of order three and our three series (2.4)–(2.6) collapse into a single series of evaluations of treble poles.

The relationship between symmetry and the relative frequency of higher order poles appearing in the asymptotics is explored in some detail in § 7.

3. Asymptotics with two internal points

Let us suppose that there are now two internal points in front of the back face, $P_1 = (m_1, n_1, p_1)$ and $P_2 = (m_2, n_2, p_2)$. Because they lie on the same side of the back face (2.2), we must have δ_1 and δ_2 both positive. With $t_2 = \rho e^{i\theta}$ and $|\theta| < \frac{1}{2}\pi$ in the integrand of (1.2) with $k = 2$, we find the logarithm of the modulus of the integrand to exhibit the dominant behaviour

$$\delta_2 \rho \cos\theta \log\rho + \mathcal{O}(\rho),$$

for large ρ. Since $\delta_2 > 0$, we see that this tends to $+\infty$ as $\rho \to \infty$ leading us to displace the t_2 integration contour to the right to obtain the asymptotic behaviour of $I(\lambda)$. Three sequences of poles are encountered in shifting the contour to the right: one each for the Γ functions present in the integrand, save $\Gamma(t_1)$ and $\Gamma(t_2)$.

The functions $\Gamma((1 - \boldsymbol{m}\cdot\boldsymbol{t})/\mu)$, $\Gamma((1 - \boldsymbol{n}\cdot\boldsymbol{t})/\nu)$ and $\Gamma((1 - \boldsymbol{p}\cdot\boldsymbol{t})/\eta)$ have their poles occurring at points

$$t_2^{(1)} = (K - m_1 t_1)/m_2, \tag{3.1}$$
$$t_2^{(2)} = (K' - n_1 t_1)/n_2,$$
$$t_2^{(3)} = (K'' - p_1 t_1)/p_2, \tag{3.2}$$

respectively, where K, K' and K'' are given in (2.7). We assume that these sequences share no points. In turn, these sequences give rise to formal asymptotic series (suppressing in each case a leading factor of $\lambda^{-1/\mu-1/\nu-1/\eta}$)

$$I_1 = \frac{1}{m_2\nu\eta}\sum_k \frac{(-1)^k}{k!}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \Gamma(t_1)\Gamma(t_2^{(1)}) \times\Gamma\left(\frac{1 - n_1 t_1 - n_2 t_2^{(1)}}{\nu}\right)\Gamma\left(\frac{1 - p_1 t_1 - p_2 t_2^{(1)}}{\eta}\right)\lambda^{-\boldsymbol{\delta}\cdot\boldsymbol{t}}\,dt_1, \tag{3.3}$$

$$I_2 = \frac{1}{\mu n_2\eta}\sum_k \frac{(-1)^k}{k!}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \Gamma(t_1)\Gamma(t_2^{(2)}) \times\Gamma\left(\frac{1 - m_1 t_1 - m_2 t_2^{(2)}}{\mu}\right)\Gamma\left(\frac{1 - p_1 t_1 - p_2 t_2^{(2)}}{\eta}\right)\lambda^{-\boldsymbol{\delta}\cdot\boldsymbol{t}}\,dt_1, \tag{3.4}$$

$$I_3 = \frac{1}{\mu\nu p_2}\sum_k \frac{(-1)^k}{k!}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \Gamma(t_1)\Gamma(t_2^{(3)}) \times\Gamma\left(\frac{1-m_1t_1-m_2t_2^{(3)}}{\mu}\right)\Gamma\left(\frac{1-n_1t_1-n_2t_2^{(3)}}{\nu}\right)\lambda^{-\boldsymbol{\delta}\cdot\boldsymbol{t}}\,dt_1, \quad (3.5)$$

with $\boldsymbol{\delta}\cdot\boldsymbol{t}$ equal to $\delta_1t_1+\delta_2t_2^{(i)}$, $i=1,2,3$, respectively. If we set $t_1=\rho e^{i\theta}$, $|\theta|<\frac{1}{2}\pi$, in each of the integrands in the series I_1, I_2 and I_3, then we find they have dominant large-ρ behaviour

$$\left.\begin{aligned}&\rho\cos\theta\log\rho\cdot(\Delta_{12}/\nu+\Delta''_{12}/\eta-M_{12})/m_2+\mathcal{O}(\rho),\\&\rho\cos\theta\log\rho\cdot(\Delta'_{12}/\eta-\Delta_{12}/\mu-N_{12})/n_2+\mathcal{O}(\rho),\\&\rho\cos\theta\log\rho\cdot(-\Delta'_{12}/\nu-\Delta''_{12}/\mu-P_{12})/p_2+\mathcal{O}(\rho),\end{aligned}\right\} \quad (3.6)$$

respectively; the various subscripted items appearing in these expressions are defined in the appendix in (A 1) and (A 3). To determine which direction to shift the integration contours in (3.3), (3.4) and (3.5), we must determine the sign of each of the parenthesized quantities following the factor $\rho\cos\theta\log\rho$. We shall see presently that the sign of each of these parenthesized quantities in (3.6) has a geometric interpretation.

Let $\overleftrightarrow{P_1P_2}$ denote the line generated by P_1 and P_2. Elementary geometry reveals that if $\overleftrightarrow{P_1P_2}$ meets the mn-, mp- and np-planes, then it does so in points with coordinates $(-\Delta''_{12}/P_{12}, -\Delta'_{12}/P_{12}, 0)$, $(-\Delta_{12}/N_{12}, 0, \Delta'_{12}/N_{12})$ and $(0, \Delta_{12}/M_{12}, \Delta''_{12}/M_{12})$, respectively. (The possibility of $\overleftrightarrow{P_1P_2}$ being parallel to a coordinate plane is discussed later.) If the parenthesized expressions in (3.6) are set to zero and rescaled by dividing through, respectively, by M_{12}, N_{12} and P_{12} then the following equations result:

$$\left.\begin{aligned}&\frac{\Delta_{12}/M_{12}}{\nu}+\frac{\Delta''_{12}/M_{12}}{\eta}-1=0,\\&\frac{-\Delta_{12}/N_{12}}{\mu}+\frac{\Delta'_{12}/N_{12}}{\eta}-1=0,\\&\frac{-\Delta''_{12}/P_{12}}{\mu}+\frac{-\Delta'_{12}/P_{12}}{\nu}-1=0.\end{aligned}\right\} \quad (3.7)$$

The first equation is equivalent to the statement that the point of intersection of $\overleftrightarrow{P_1P_2}$ with the np-plane lies on the line of intersection of the back face of the Newton diagram with the np-plane. The second and third equations, similarly, assert the presence on the line of intersection of the back face with either of the mp- or mn-planes of the point of intersection of $\overleftrightarrow{P_1P_2}$ with each of these coordinate planes, respectively.

The determination of the sign of the parenthesized quantities in (3.6) is therefore equivalent to deciding on which side of the line of intersection of the back face with a coordinate plane a point of intersection of $\overleftrightarrow{P_1P_2}$ happens to lie. The difficulty in determining this stems from the fact that the distribution in space of P_1 and P_2 can lead to M_{12}, N_{12} and P_{12} being of either sign. Let us therefore fix the sign of these quantities so as to permit further analysis: let us suppose that $\overleftrightarrow{P_1P_2}$ punctures the mn- and np-planes in the positive octant, and that $M_{12}>0$, $P_{12}<0$. This setting is depicted in figure 2, with a view assuming the observer is behind the back face, looking towards the origin.

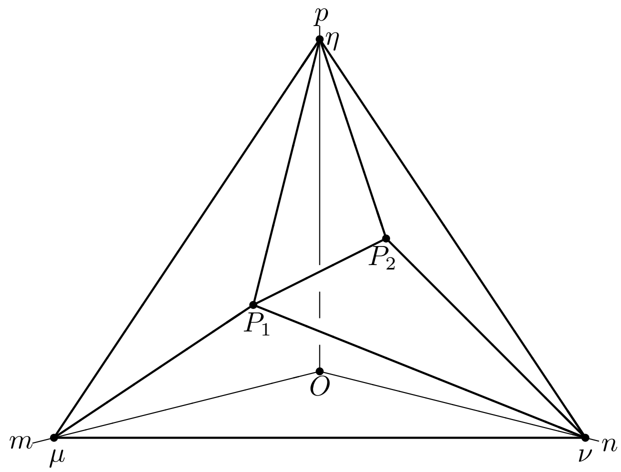

Figure 2. Newton diagram with two internal points.

Because $M_{12} > 0$, $P_{12} < 0$ and $\overleftrightarrow{P_1P_2}$ meets the mn- and np-planes in the positive octant, we may deduce that $\Delta_{12} > 0$ and $\Delta''_{12} > 0$. Consequently, the statement that the point of intersection of $\overleftrightarrow{P_1P_2}$ with the np-plane lie below the line of intersection of the back face with that coordinate plane is equivalent to the inequality

$$\frac{\Delta_{12}/M_{12}}{\nu} + \frac{\Delta''_{12}/M_{12}}{\eta} - 1 < 0,$$

and the equivalent statement involving the point of intersection of $\overleftrightarrow{P_1P_2}$ and the mn-plane results in

$$\frac{-\Delta''_{12}/P_{12}}{\mu} + \frac{-\Delta'_{12}/P_{12}}{\nu} - 1 < 0.$$

Both of these inequalities follow from the assumption of a 'convex' configuration for the internal points P_1 and P_2: one needs only to form the plane $\Pi(O, P_1, P_2)$ and intersect it with the Newton diagram to see this.

The second equation in (3.7) on first glance appears to be redundant, but in fact captures additional convexity information regarding the Newton diagram. This second equation is a statement regarding the location on the mp-plane of the point $(-\Delta_{12}/N_{12}, 0, \Delta'_{12}/N_{12})$, the point of intersection of the mp-plane with the line $\overleftrightarrow{P_1P_2}$. If $N_{12} > 0$, then this intersection point lies behind $(m < 0)$ the np-plane and above $(p > 0)$ the mn-plane. If $N_{12} < 0$, then the intersection point lies below $(p < 0)$ the mn-plane, but in front $(m > 0)$ of the np-plane. Where it lies relative to the back face (2.2) is determined by the sign of the left-hand side of the second equation of (3.7).

Consider the plane $\Pi(O, P_1, P_2)$ generated by the origin and the internal points P_1 and P_2. If $(-\Delta_{12}/N_{12}, 0, \Delta'_{12}/N_{12})$ does not satisfy $m/\mu + p/\eta - 1 < 0$, then $(-\Delta_{12}/N_{12}, 0, \Delta'_{12}/N_{12})$ lies on the back face or in the half-space determined by the back face not containing the origin. Suppose, for the purpose of illustration, that $(-\Delta_{12}/N_{12})/\mu + (\Delta'_{12}/N_{12})/\eta - 1 > 0$. Then, in the plane $\Pi(O, P_1, P_2)$, we see that the intersection point $(-\Delta_{12}/N_{12}, 0, \Delta'_{12}/N_{12})$ is positioned in such a way that P_1 cannot be a vertex of the Newton diagram.

This situation is depicted in figure 3, with the line of intersection of $\Pi(O, P_1, P_2)$ and the mp-plane corresponding to $\overleftrightarrow{OA}$, the intersection of $\overleftrightarrow{P_1P_2}$ and the mp-plane

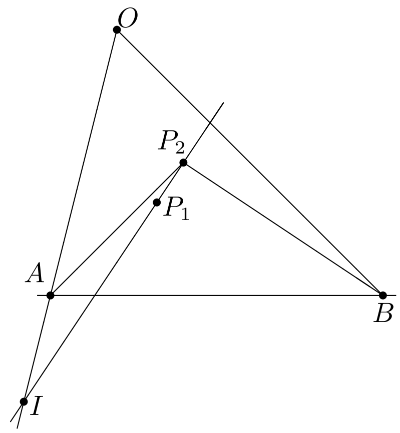

Figure 3. Planar section $\Pi(O, P_1, P_2)$ of the Newton diagram in the case where the intersection point $I = (-\Delta_{12}/N_{12}, 0, \Delta'_{12}/N_{12})$ lies on the side of the back face not containing the origin.

labelled as I, the line of intersection of the back face with the plane $\Pi(O, P_1, P_2)$ appearing as $\overleftrightarrow{AB}$ and the line $\overleftrightarrow{OB}$ corresponding to the line of intersection of $\Pi(O, P_1, P_2)$ with one of the other coordinate planes (either the mn- or np-plane).

Because the convex hull used in the construction of the Newton diagram is convex (recall (I, § 2) and the half-space defined by the back face containing the origin is convex, their intersection must be a bounded convex solid. By intersecting this with the plane $\Pi(O, P_1, P_2)$, a convex planar region must result. This region is $\triangle ABP_2$, and since we have assumed that the intersection point I lies on the side of the back face not containing the origin, it follows that P_1 lies in the interior of $\triangle ABP_2$, a violation of the hypothesis that P_1 and P_2 be vertices of the Newton diagram.

If instead we assume that the intersection point I lies on the back face, i.e. $m/\mu + p/\eta - 1 = 0$ is satisfied by I, then a planar section through P_1 and P_2 will reveal that P_1 lies on the line segment $\overline{AP_2}$. In this circumstance, the hypothesis that P_1 and P_2 both be vertices fails: P_1 will be a point of the Newton diagram, but not an extreme point.

If the intersection point I lies on the same side of the back face as the origin, i.e., $m/\mu + p/\eta - 1 < 0$ is satisfied by I, then the geometry of the planar section of the Newton diagram changes. The result, depicted in figure 4, reveals the planar section of the bounded convex region formed from the convex hull, used to construct the Newton diagram, intersected with the half-space defined by the back face containing the origin; this polygonal region has vertices ABP_2P_1, in the notation of the previous paragraphs.

We see, therefore, that regardless of the sign of N_{12}, we must have the intersection point I and the origin in the same half-space determined by the back face. Convexity then permits us to assert the following inequalities:

$$\left.\begin{aligned} \frac{\Delta_{12}/M_{12}}{\nu} + \frac{\Delta''_{12}/M_{12}}{\eta} - 1 < 0, \\ \frac{-\Delta_{12}/N_{12}}{\mu} + \frac{\Delta'_{12}/N_{12}}{\eta} - 1 < 0, \\ \frac{-\Delta''_{12}/P_{12}}{\mu} + \frac{-\Delta'_{12}/P_{12}}{\nu} - 1 < 0. \end{aligned}\right\} \qquad (3.8)$$

With these inequalities at hand, we can determine the large-ρ behaviour of each of

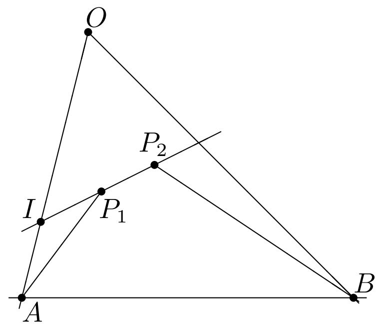

Figure 4. Planar section $\Pi(O, P_1, P_2)$ of the Newton diagram in the case where the intersection point $I = (-\Delta_{12}/N_{12}, 0, \Delta'_{12}/N_{12})$ lies on the side of the back face containing the origin.

the estimates in (3.6), and so determine in which direction the t_1 integration contours in (3.3)–(3.5) must be shifted in order to extract the asymptotic behaviour of each of the integrals in those formal series. Let us assume that $N_{12} < 0$, so that our assumptions on the distribution of P_1 and P_2 in space become

$$M_{12} > 0, \quad N_{12} < 0, \quad P_{12} < 0. \tag{3.9}$$

Returning to the order estimates (3.6), we can now assert that the first estimate tends to $-\infty$ and the second and third to $+\infty$ as $\rho \to \infty$. Accordingly, the t_1 contours in the integrals in I_1, I_2 and I_3 of (3.3)–(3.5) must be shifted left, right and right, respectively, to determine their asymptotic behaviour.

In displacing the t_1 contour in (3.3) to the left, we encounter poles of the integrand at $t_1^{(1)} = -l$ from the factor $\Gamma(t_1)$ in the integrand of (3.3) and at

$$t_1^{(2)} = (n_2K - m_2L')/\Delta_{12} \leqslant 0, \tag{3.10}$$

$$t_1^{(3)} = (p_2K - m_2L'')/\Delta''_{12} \leqslant 0, \tag{3.11}$$

from the factors $\Gamma((1 - n_1t_1 - n_2t_2^{(1)})/\nu)$ and $\Gamma((1 - p_1t_1 - p_2t_2^{(1)})/\eta)$, respectively. The parameter l is a non-negative integer, and we have set

$$L = 1 + \mu l, \quad L' = 1 + \nu l, \quad L'' = 1 + \eta l; \tag{3.12}$$

recall (2.7). The factor $\Gamma(t_2^{(1)})$ encounters no poles as the t_1 contour is shifted left.

The $t_1^{(1)}$ sequence of poles gives rise to the formal asymptotic series (suppressing the leading factor of $\lambda^{-1/\mu-1/\nu-1/\eta}$)

$$I_{11} \equiv I_{m_2\nu\eta} = \frac{1}{m_2\nu\eta}\sum_{k,l} f_{11}(k,l), \tag{3.13}$$

where

$$f_{11}(k,l) = \frac{(-1)^{k+l}}{k!\,l!}\Gamma\left(\frac{K + m_1l}{m_2}\right)\Gamma\left(\frac{m_2 - n_2K - \Delta_{12}l}{m_2\nu}\right) \times\Gamma\left(\frac{m_2 - p_2K - \Delta''_{12}l}{m_2\eta}\right)\lambda^{-\boldsymbol{\delta}\cdot\boldsymbol{t}},$$

with

$$-\boldsymbol{\delta}\cdot\boldsymbol{t} = -\delta_1t_1^{(1)} - \delta_2t_2^{(1)}(t_1^{(1)}) = -\frac{\delta_2K}{m_2} + \left(\frac{\Delta_{12}}{\nu} + \frac{\Delta''_{12}}{\eta} - M_{12}\right)\frac{l}{m_2}.$$

Similarly, we obtain from the $t_1^{(2)}$ and $t_1^{(3)}$ sequences the formal asymptotic series

$$I_{12} = \frac{1}{\eta \Delta_{12}} \sum_{k,l'} f_{12}(k,l), \tag{3.14}$$

$$I_{13} = \frac{1}{\nu \Delta''_{12}} \sum_{k,l'} f_{13}(k,l), \tag{3.15}$$

where

$$f_{12}(k,l) = \frac{(-1)^{k+l}}{k!l!} \Gamma\left(\frac{-n_1 K + m_1 L'}{\Delta_{12}}\right) \Gamma\left(\frac{n_2 K - m_2 L'}{\Delta_{12}}\right) \times \Gamma\left(\frac{\Delta_{12} + \Delta'_{12} K - \Delta''_{12} L'}{\eta \Delta_{12}}\right) \lambda^{-\boldsymbol{\delta}\cdot\boldsymbol{t}}, \tag{3.16}$$

with

$$-\boldsymbol{\delta}\cdot\boldsymbol{t} = \left(\frac{\Delta_{12}}{\mu} - \frac{\Delta'_{12}}{\eta} + N_{12}\right)\frac{K}{\Delta_{12}} + \left(\frac{\Delta_{12}}{\nu} + \frac{\Delta''_{12}}{\eta} - M_{12}\right)\frac{L'}{\Delta_{12}}, \tag{3.17}$$

and

$$f_{13}(k,l) = \frac{(-1)^{k+l}}{k!l!} \Gamma\left(\frac{p_2 K - m_2 L''}{\Delta''_{12}}\right) \Gamma\left(\frac{-p_1 K + m_1 L''}{\Delta''_{12}}\right) \times \Gamma\left(\frac{\Delta''_{12} - \Delta'_{12} K - \Delta_{12} L''}{\nu \Delta''_{12}}\right) \lambda^{-\boldsymbol{\delta}\cdot\boldsymbol{t}},$$

with

$$-\boldsymbol{\delta}\cdot\boldsymbol{t} = \left(\frac{\Delta''_{12}}{\mu} + \frac{\Delta'_{12}}{\nu} + P_{12}\right)\frac{K}{\Delta''_{12}} + \left(\frac{\Delta_{12}}{\nu} + \frac{\Delta''_{12}}{\eta} - M_{12}\right)\frac{L''}{\Delta''_{12}}.$$

The primes appearing on the subscripts in the sums (3.14) and (3.15) indicate that the summation index l in each case is subject to a constraint: for I_{12}, this is that $t_1^{(2)} \leqslant 0$, and for I_{13}, that $t_1^{(3)} \leqslant 0$; recall (3.10) and (3.11).

For I_2 in (3.4) we must displace the integration contour to the right, and in so doing, encounter poles of the integrands at

$$t_1^{(1)} = (K' + n_2 l)/n_1,$$

$$t_1^{(2)} = (-m_2 K' + n_2 L)/\Delta_{12} > 0, \tag{3.18}$$

$$t_1^{(3)} = (p_2 K' - n_2 L'')/\Delta'_{12} > 0. \tag{3.19}$$

Points in the $t_1^{(1)}$ sequence give rise to the series of contributions (with the usual suppression of $\lambda^{-1/\mu-1/\nu-1/\eta}$)

$$I_{21} \equiv I_{\mu n_1 \eta} = \frac{1}{\mu n_1 \eta} \sum_{k,l} f_{21}(k,l),$$

where

$$f_{21}(k,l) = \frac{(-1)^{k+l}}{k!l!} \Gamma\left(\frac{K' + n_2 l}{n_1}\right) \Gamma\left(\frac{n_1 - m_1 K' - \Delta_{12} l}{\mu n_1}\right) \times \Gamma(\frac{n_1 - p_1 K' + \Delta'_{12} l}{n_1 \eta}) \lambda^{-\boldsymbol{\delta}\cdot\boldsymbol{t}},$$

with

$$-\boldsymbol{\delta}\cdot\boldsymbol{t} = -\delta_1 t_1^{(1)} - \delta_2 t_2^{(2)}(t_1^{(1)}) = -\frac{\delta_1 K'}{n_1} + \left(\frac{\Delta_{12}}{\mu} - \frac{\Delta'_{12}}{\eta} + N_{12}\right)\frac{l}{n_1}.$$

Similarly, we obtain from the $t_1^{(2)}$ and $t_1^{(3)}$ sequences the formal asymptotic series

$$I_{22} = \frac{1}{\eta\Delta_{12}}\sum_{k,l'} f_{12}(l,k), \tag{3.20}$$

$$I_{23} = -\frac{1}{\mu\Delta'_{12}}\sum_{k,l'} f_{23}(k,l),$$

where

$$f_{23}(k,l) = \frac{(-1)^{k+l}}{k!l!}\Gamma\left(\frac{p_2K' - n_2L''}{\Delta'_{12}}\right)\Gamma\left(\frac{-p_1K' + n_1L''}{\Delta'_{12}}\right) \times\Gamma\left(\frac{\Delta'_{12} - \Delta''_{12}K' + \Delta_{12}L''}{\mu\Delta'_{12}}\right)\lambda^{-\boldsymbol{\delta}\cdot\boldsymbol{t}},$$

with

$$-\boldsymbol{\delta}\cdot\boldsymbol{t} = \left(\frac{\Delta''_{12}}{\mu} + \frac{\Delta'_{12}}{\nu} + P_{12}\right)\frac{K'}{\Delta'_{12}} + \left(-\frac{\Delta_{12}}{\mu} + \frac{\Delta'_{12}}{\eta} - N_{12}\right)\frac{L''}{\Delta'_{12}}. \tag{3.21}$$

As before, for I_{12} and I_{13}, the prime on a summation index indicates a restriction. In the cases of I_{22} and I_{23}, these are $t_1^{(2)} > 0$ and $t_1^{(3)} > 0$, respectively; recall (3.18) and (3.19). When computing $f_{12}(l,k)$ in (3.20), it is worth noting that replacing k by l and vice versa in (3.16) is equivalent to replacing K by L and K' by L'.

For I_3 in (3.5) we shift the integration contour to the right as for I_2, and encounter poles of the integrands at

$$t_1^{(1)} = (K'' + p_2 l)/p_1,$$

$$t_1^{(2)} = (-m_2K'' + p_2L)/\Delta''_{12} > 0, \tag{3.22}$$

$$t_1^{(3)} = (-n_2K'' + p_2L')/\Delta'_{12} > 0. \tag{3.23}$$

The poles in the $t_1^{(1)}$ sequence give rise to the series of contributions (with the usual suppression of $\lambda^{-1/\mu-1/\nu-1/\eta}$)

$$I_{31} \equiv I_{\mu\nu p_1} = \frac{1}{\mu\nu p_1}\sum_{k,l} f_{31}(k,l), \tag{3.24}$$

where

$$f_{31}(k,l) = \frac{(-1)^{k+l}}{k!l!}\Gamma\left(\frac{K'' + p_2 l}{p_1}\right)\Gamma\left(\frac{p_1 - m_1K'' - \Delta''_{12}l}{\mu p_1}\right) \times\Gamma\left(\frac{p_1 - n_1K'' - \Delta'_{12}l}{\nu p_1}\right)\lambda^{-\boldsymbol{\delta}\cdot\boldsymbol{t}},$$

with

$$-\boldsymbol{\delta}\cdot\boldsymbol{t} = -\delta_1 t_1^{(1)} - \delta_2 t_2^{(3)}(t_1^{(1)}) = -\frac{\delta_1 K''}{p_1} + \left(\frac{\Delta''_{12}}{\mu} + \frac{\Delta'_{12}}{\nu} + P_{12}\right)\frac{l}{p_1}.$$

The $t_1^{(2)}$ and $t_1^{(3)}$ sequences give rise to the formal asymptotic series

$$I_{32} = \frac{1}{\nu \Delta''_{12}} \sum_{k,l'} f_{13}(l,k), \tag{3.25}$$

$$I_{33} = \frac{1}{\mu \Delta'_{12}} \sum_{k,l'} f_{23}(l,k), \tag{3.26}$$

where the primes on the summation indices indicate the restrictions $t_1^{(2)} > 0$ for I_{32} and $t_1^{(3)} > 0$ for I_{33}; recall (3.22) and (3.23).

We can now assemble the expansions I_{11}, I_{12}, I_{13}, I_{21}, I_{22}, I_{23}, I_{31}, I_{32} and I_{33} into an expansion of (1.1) with $k = 2$ and $c_1 = c_2 = 1$. Before doing so, we observe that some simplification is possible. Observe that the series I_{12} and I_{22} are of the same form. If we perform the change of summation indices $k \to l$ and $l \to k$ in (3.20), then both series for I_{12} and I_{22} have precisely the same form, albeit with different restrictions imposed on their summation indices. For I_{22}, the restriction governing its indices was $t_1^{(2)} > 0$, or

$$(-m_2 K' + n_2 L)/\Delta_{12} > 0;$$

recall (3.18). After the change of variables $k \to l$, $l \to k$, K' will become L' and L will become K and the restriction assumes the form

$$(n_2 K - m_2 L')/\Delta_{12} > 0.$$

However, from (3.10), we see that the inequality governing I_{12} is

$$(n_2 K - m_2 L')/\Delta_{12} \leqslant 0,$$

the complement of the inequality governing our transformed I_{22}. Therefore, I_{12} and I_{22} may be fused into a single series, $I_{\eta\Delta_{12}}$ say, with no restrictions placed on the non-negative summation indices k and l:

$$I_{\eta\Delta_{12}} = \frac{1}{\eta \Delta_{12}} \sum_{k,l} f_{12}(k,l).$$

In a similar fashion, we see that, apart from a minus sign attached to I_{23}, the series I_{23} and I_{33} are of the same form. If we perform the change of summation indices $k \to l$ and $l \to k$ on I_{33}, then the resulting series has the same summand, $f_{23}(k,l)$, as I_{23}, and the inequality restricting I_{33}, $t_1^{(3)} > 0$ or

$$(-n_2 K'' + p_2 L')/\Delta'_{12} > 0,$$

appearing in (3.23), becomes

$$(p_2 K' - n_2 L'')/\Delta'_{12} > 0.$$

This is identical to the restriction governing I_{23}, displayed in (3.19), and so the series I_{23} and I_{33} annihilate each other.

Finally, the two series I_{13} and I_{32} are similar in form. By reasoning as we have just done, we find that these two series fuse into a single series, which we denote $I_{\nu\Delta''_{12}}$, identical to I_{13} and I_{32} but with no restriction placed on the non-negative integral summation indices k and l:

$$I_{\nu\Delta''_{12}} = \frac{1}{\nu \Delta''_{12}} \sum_{k,l} f_{13}(k,l).$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Collecting together $I_{m_2\nu\eta}$, $I_{\mu n_1\eta}$, $I_{\mu\nu p_1}$, $I_{\eta\Delta_{12}}$ and $I_{\nu\Delta''_{12}}$ and restoring the factor $\lambda^{-1/\mu-1/\nu-1/\eta}$, we arrive at the large-λ expansion

$$I(\lambda) \sim \lambda^{-1/\mu-1/\nu-1/\eta}(I_{m_2\nu\eta} + I_{\mu n_1\eta} + I_{\mu\nu p_1} + I_{\eta\Delta_{12}} + I_{\nu\Delta''_{12}}). \tag{3.27}$$

The correspondence between the series in this compound expansion and the faces of the Newton diagram is easily seen (as indicated by the subscripts used in the series on the right-hand side): $I_{m_2\nu\eta}$ corresponds to the face with vertices P_2, $(0,\nu,0)$ and $(0,0,\eta)$; $I_{\mu n_1\eta}$ corresponds to the face with vertices $(\mu,0,0)$, P_1 and $(0,0,\eta)$; $I_{\mu\nu p_1}$ corresponds to the face with vertices $(\mu,0,0)$, $(0,\nu,0)$ and P_1; $I_{\eta\Delta_{12}}$ corresponds to the face with vertices P_1, P_2 and $(0,0,\eta)$; and finally, $I_{\nu\Delta''_{12}}$ corresponds to the face with vertices P_1, P_2 and $(0,\nu,0)$. The reader may want to review the arrangement of faces by referring to figure 2.

Were we to assume that $N_{12} > 0$ (recall (3.9)), then a compound asymptotic expansion with five constituent series would again result: there would still be the three series corresponding to the three faces of the Newton diagram each with two vertices anchored on coordinate axes, and the series $I_{\nu\Delta''_{12}}$ would remain, but instead of the series $I_{\eta\Delta_{12}}$ we would find the series $I_{\mu\Delta'_{12}}$. The computation would proceed as before but instead of shifting the t_1 integration contours in I_2 to the right, we would be displacing them to the left. The resulting asymptotic expansion would have the form

$$I(\lambda) \sim \lambda^{-1/\mu-1/\nu-1/\eta}(I_{m_2\nu\eta} + I_{\mu n_2\eta} + I_{\mu\nu p_1} + I_{\mu\Delta'_{12}} + I_{\nu\Delta''_{12}}).$$

In this setting we note that the edge counts for the vertices P_1 and P_2 would be reversed: P_1 would have an edge count of three, whilst P_2 would have an edge count of four.

If any one of the order estimates in (3.6) were to vanish, then a finer estimate of the growth of the logarithm of the modulus of the integrand would have to be performed in order to determine the direction in which an integration contour would have to be displaced. In addition, if a face of the Newton diagram were a quadrilateral instead of a triangle, then this process would break down. A discussion of the case of non-triangular faces in the Newton diagram is provided in § 6.

4. Asymptotics with three internal points

The final case we treat is one in which a face of the Newton diagram does not make contact with any of the m, n or p coordinate axes. The simplest such case occurs for three internal points, although we mention that there are configurations with three internal points where every (triangular) face makes contact with a coordinate axis. It is in the case of a triangular face isolated from the coordinate axes that the analogue of 'relative convexity' for Newton diagrams for two-dimensional phases appears; (cf. I, § 6.1).

To fix values for this investigation, let us assume that the three internal points are arranged as in figure 5. There are a total of seven faces to the Newton diagram, $F_1, \ldots, F_7$, with vertices given in an order consistent with an orientation selected by the normal to each face pointing to the back face:

$F_1 : (\mu,0,0), P_1, P_3,$ $\quad$ $F_5 : (\mu,0,0), (0,\nu,0), P_1,$

$F_2 : (\mu,0,0), P_3, (0,0,\eta),$ $\quad$ $F_6 : (0,\nu,0), P_2, P_1,$

$F_3 : (0,0,\eta), P_3, P_2,$ $\quad$ $F_7 : (0,\nu,0), (0,0,\eta), P_2.$

$F_4 : P_1, P_2, P_3,$

With this ordering and orientation, writing $\det F_i$ for the determinant of the array formed by writing the vertices as rows of the array, we have

$$\left.\begin{array}{ll} \det F_1 = \mu\Delta'_{13} > 0, & \det F_5 = \mu\nu p_1 > 0, \\ \det F_2 = \mu n_3\eta > 0, & \det F_6 = \nu\Delta''_{23}, \\ \det F_3 = \eta\Delta_{23} > 0, & \det F_7 = m_2\nu\eta > 0, \\ \multicolumn{2}{c}{\det F_4 = \Delta_{123},} \end{array}\right\} \tag{4.1}$$

recall the definitions provided in (A 1) and (A 2). We shall impose the additional constraints

$$M_{12} > 0, \quad M_{23} < 0, \quad N_{13} > 0, \quad N_{23} > 0, \quad P_{12} < 0, \quad P_{13} < 0, \quad P_{23} > 0, \tag{4.2}$$

which parallel the role of (3.9) of the previous section.

We set $k = 3$ in (1.1) and (1.2) and begin by displacing the t_3 contour. With $t_3 = \rho e^{i\theta}$, $|\theta| < \frac{1}{2}\pi$, we find that the logarithm of the modulus of the integrand has the asymptotic behaviour

$$\delta_3\rho\cos\theta\log\rho + \mathcal{O}(\rho),$$

and with $\delta_3 > 0$, i.e. P_3 is in front of the back face, we see that this estimate tends to ∞ as $\rho \to \infty$. Therefore, we shift the t_3 contour to the right to obtain the asymptotic behaviour of $I(\lambda)$. In moving the contour to the right, we encounter poles in the t_3-plane at

$$t_3^{(1)} = (K - m_1t_1 - m_2t_2)/m_3, \quad t_3^{(2)} = (K' - n_1t_1 - n_2t_2)/n_3,$$
$$t_3^{(3)} = (K'' - p_1t_1 - p_2t_2)/p_3,$$

where k is a non-negative integer, and the quantities K, K' and K'' are given in (2.7).

Poles from the $t_3^{(1)}$ sequence give rise to the series of contributions (suppressing a factor $\lambda^{-1/\mu-1/\nu-1/\eta}$)

$$\begin{aligned} I_1 = \frac{1}{m_3\nu\eta}\sum_k \frac{(-1)^k}{k!}\left(\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\right)^2 \\ \times\Gamma(t_1)\Gamma(t_2)\Gamma\left(\frac{K - m_1t_1 - m_2t_2}{m_3}\right)\Gamma\left(\frac{m_3 - n_3K - \Delta_{13}t_1 - \Delta_{23}t_2}{m_3\nu}\right) \\ \times\Gamma\left(\frac{m_3 - p_3K + \Delta''_{13}t_1 - \Delta''_{23}t_2}{m_3\eta}\right)\lambda^{-\boldsymbol{\delta}\cdot\boldsymbol{t}}\,dt_1dt_2, \end{aligned} \tag{4.3}$$

where the subscripted Δ are defined in (A 1) and $\boldsymbol{\delta}\cdot\boldsymbol{t} = \delta_1t_1 + \delta_2t_2 + \delta_3t_3^{(1)}$. Similarly, poles from the $t_3^{(2)}$ and $t_3^{(3)}$ sequences give rise to the formal series

$$\begin{aligned} I_2 = \frac{1}{\mu n_3\eta}\sum_k \frac{(-1)^k}{k!}\left(\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\right)^2 \\ \times\Gamma(t_1)\Gamma(t_2)\Gamma\left(\frac{K' - n_1t_1 - n_2t_2}{n_3}\right)\Gamma\left(\frac{n_3 - m_3K' + \Delta_{13}t_1 + \Delta_{23}t_2}{\mu n_3}\right) \\ \times\Gamma\left(\frac{n_3 - p_3K' + \Delta'_{13}t_1 + \Delta'_{23}t_2}{n_3\eta}\right)\lambda^{-\boldsymbol{\delta}\cdot\boldsymbol{t}}\,dt_1dt_2, \end{aligned} \tag{4.4}$$

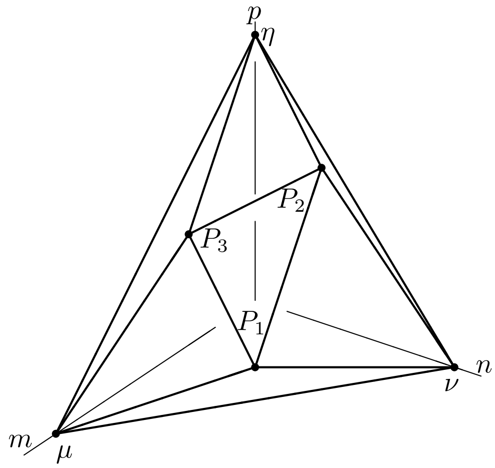

Figure 5. Geometry of the Newton diagram for three internal points forming a triangular face not touching the coordinate axes. The view is from the side of the Newton diagram not containing the origin.

and

$$I_3 = \frac{1}{\mu\nu p_3}\sum_k \frac{(-1)^k}{k!}\left(\frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\right)^2 \times\Gamma(t_1)\Gamma(t_2)\Gamma\left(\frac{K''-p_1t_1-p_2t_2}{p_3}\right)\Gamma\left(\frac{p_3-m_3K''-\Delta''_{13}t_1+\Delta''_{23}t_2}{\mu p_3}\right) \times\Gamma\left(\frac{p_3-n_3K''-\Delta'_{13}t_1-\Delta'_{23}t_2}{\nu p_3}\right)\lambda^{-\boldsymbol{\delta}\cdot\boldsymbol{t}}\,\mathrm{d}t_1\mathrm{d}t_2, \tag{4.5}$$

with $\boldsymbol{\delta}\cdot\boldsymbol{t}$ in each of (4.4) and (4.5) given, respectively, by $\delta_1t_1+\delta_2t_2+\delta_3t_3^{(2)}$ and $\delta_1t_1+\delta_2t_2+\delta_3t_3^{(3)}$.

For the integrals in each of the series (4.3)–(4.5), we find, putting $t_2=\rho\mathrm{e}^{\mathrm{i}\theta}$, $|\theta|<\frac{1}{2}\pi$, the dominant large-ρ behaviour of the logarithm of the modulus of each integrand to

$$\left.\begin{aligned} &-\rho\cos\theta\log\rho\cdot\frac{1}{m_3}\left(\frac{\Delta_{23}}{\nu}+\frac{\Delta''_{23}}{\eta}+M_{23}\right)+\mathcal{O}(\rho),\\ &\rho\cos\theta\log\rho\cdot\frac{1}{n_3}\left(\frac{\Delta_{23}}{\mu}+\frac{\Delta'_{23}}{\eta}-N_{23}\right)+\mathcal{O}(\rho),\\ &\rho\cos\theta\log\rho\cdot\frac{1}{p_3}\left(\frac{\Delta''_{23}}{\mu}-\frac{\Delta'_{23}}{\nu}-P_{23}\right)+\mathcal{O}(\rho), \end{aligned}\right\} \tag{4.6}$$

respectively. Convexity criteria involving the points of intersection of $\overleftrightarrow{P_iP_j}$ with the coordinate planes require that

$$\frac{(-\Delta_{23}/M_{23})}{\nu}+\frac{(-\Delta''_{23}/M_{23})}{\eta}-1<0,$$
$$\frac{(\Delta_{23}/N_{23})}{\mu}+\frac{(\Delta'_{23}/N_{23})}{\eta}-1<0,$$
$$\frac{(\Delta''_{23}/P_{23})}{\mu}+\frac{(-\Delta'_{23}/P_{23})}{\nu}-1<0,$$

as discussed in the previous section following (3.6) and prior to (3.10). In view of the additional constraints (4.2), these can be recast as

$$\frac{\Delta_{23}}{\nu}+\frac{\Delta''_{23}}{\eta}+M_{23}<0,\qquad \frac{\Delta_{23}}{\mu}+\frac{\Delta'_{23}}{\eta}-N_{23}<0,\qquad \frac{\Delta''_{23}}{\mu}-\frac{\Delta'_{23}}{\nu}-P_{23}<0.$$

With these estimates, we see that the order estimates in (4.6) tend to, respectively, $+\infty$, $-\infty$ and $-\infty$ as $\rho\to\infty$, which in turn indicates that, for an asymptotic evaluation, the t_2 integration contours in I_1, I_2 and I_3 be shifted, respectively, to the right, the left and the left.

(*a*) *Treatment of the series of integrals I_1*

In displacing the t_2 contour in I_1 to the right, we encounter poles at the points

$$t_2^{(1)}=(K+m_3l-m_1t_1)/m_2,$$
$$t_2^{(2)}=(-n_3K+m_3L'-\Delta_{13}t_1)/\Delta_{23}>0, \tag{4.7}$$
$$t_2^{(3)}=(-p_3K+m_3L''+\Delta''_{13}t_1)/\Delta''_{23}>0, \tag{4.8}$$

where l is a non-negative integer and the K and L are given in (2.7) and (3.12). The $t_2^{(1)}$ sequence gives rise to the formal series

$$I_{11}=\frac{1}{m_2\nu\eta}\sum_{k,l}\frac{(-1)^{k+l}}{k!l!}\frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\Gamma(t_1)\Gamma\left(\frac{K+m_3l-m_1t_1}{m_2}\right)$$
$$\times\Gamma\left(\frac{m_2-n_2K-\Delta_{23}l+\Delta_{12}t_1}{m_2\nu}\right)$$
$$\times\Gamma\left(\frac{m_2-p_2K-\Delta''_{23}l+\Delta''_{12}t_1}{m_2\eta}\right)\lambda^{-\boldsymbol{\delta}\cdot\boldsymbol{t}}\,\mathrm{d}t_1, \tag{4.9}$$

where, of course, $\boldsymbol{\delta}\cdot\boldsymbol{t}=\delta_1t_1+\delta_2t_2^{(1)})+\delta_3t_3^{(1)}(t_2^{(1)})$. With $t_1=\rho\mathrm{e}^{\mathrm{i}\theta}$, $|\theta|<\frac{1}{2}\pi$, we find that the logarithm of the modulus of the integrands in (4.9) have the large-ρ behaviour

$$\rho\cos\theta\log\rho\cdot\left(\frac{\Delta_{12}}{\nu}+\frac{\Delta''_{12}}{\eta}-M_{12}\right)/m_2+\mathcal{O}(\rho).$$

Since $M_{12}>0$ (recall (4.2)) and since $\overleftrightarrow{P_1P_2}$ meets the np-plane below the line of intersection of the back face with the np-plane, we know that $(\Delta_{12}/M_{12})/\nu+(\Delta''_{12}/M_{12})/\eta-1<0$. Thus, this order estimate tends to $-\infty$ as $\rho\to\infty$ and so we must displace the t_1 integration contour to the left.

In shifting t_1 to the left, we encounter poles of the integrands at

$$t_1^{(1)}=-r,$$
$$t_1^{(2)}=(n_2K+\Delta_{23}l-m_2R')/\Delta_{12}\leqslant 0, \tag{4.10}$$
$$t_1^{(3)}=(p_2K+\Delta''_{23}l-m_2R'')/\Delta''_{12}\leqslant 0, \tag{4.11}$$

where r is a non-negative integer, and the parameters R, R' and R'' are defined by

$$R=1+\mu r,\quad R'=1+\nu r,\quad R''=1+\eta r. \tag{4.12}$$

The sequence of poles $t_1^{(1)}$ gives rise to the contribution (as usual, suppressing a leading factor of $\lambda^{-1/\mu-1/\nu-1/\eta}$)

$$I_{m_2\nu\eta} \equiv I_{111} = \frac{1}{m_2\nu\eta}\sum_{k,l,r} f_{111}(k,l,r), \tag{4.13}$$

where

$$f_{111}(k,l,r) = \frac{(-1)^{k+l+r}}{k!l!r!}\Gamma\left(\frac{K+m_3l+m_1r}{m_2}\right)\Gamma\left(\frac{m_2-n_2K-\Delta_{23}l-\Delta_{12}r}{m_2\nu}\right) \times\Gamma\left(\frac{m_2-p_2K-\Delta''_{23}l-\Delta''_{12}r}{m_2\eta}\right)\lambda^{-\boldsymbol{\delta}\cdot\boldsymbol{t}},$$

with

$$-\boldsymbol{\delta}\cdot\boldsymbol{t} = -\delta_1 t_1^{(1)} - \delta_2 t_2^{(1)}(t_1^{(1)}) - \delta_3 t_3^{(1)}(t_1^{(1)}, t_2^{(1)}) = -\frac{\delta_2}{m_2}K + \left(\frac{\Delta_{23}}{\nu} + \frac{\Delta''_{23}}{\eta} + M_{23}\right)\frac{l}{m_2} + \left(\frac{\Delta_{12}}{\nu} + \frac{\Delta''_{12}}{\eta} - M_{12}\right)\frac{r}{m_2}.$$

In similar fashion, we find that the $t_1^{(2)}$ sequence yields the sum

$$I_{112} = \frac{1}{\eta\Delta_{12}}\sum_{k,l,r'} f_{112}(k,l,r), \tag{4.14}$$

where

$$f_{112}(k,l,r) = \frac{(-1)^{k+l+r}}{k!l!r!}\Gamma\left(\frac{n_2K+\Delta_{23}l-m_2R'}{\Delta_{12}}\right)\Gamma\left(\frac{-n_1K-\Delta_{13}l+m_1R'}{\Delta_{12}}\right) \times\Gamma\left(\frac{\Delta_{12}+\Delta'_{12}K+\Delta_{123}l-\Delta''_{12}R'}{\eta\Delta_{12}}\right)\lambda^{-\boldsymbol{\delta}\cdot\boldsymbol{t}},$$

with

$$-\boldsymbol{\delta}\cdot\boldsymbol{t} = \left(\frac{\Delta_{12}}{\mu} - \frac{\Delta'_{12}}{\eta} + N_{12}\right)\frac{K}{\Delta_{12}} + \left(-\frac{\Delta_{123}}{\eta} + \Delta_{12} + \Delta_{13} - \Delta_{23}\right)\frac{l}{\Delta_{12}} + \left(\frac{\Delta_{12}}{\nu} + \frac{\Delta''_{12}}{\eta} - M_{12}\right)\frac{R''}{\Delta_{12}},$$

and the $t_1^{(3)}$ sequence produces the series

$$I_{113} = \frac{1}{\nu\Delta''_{12}}\sum_{k,l,r'} f_{113}(k,l,r), \tag{4.15}$$

where

$$f_{113}(k,l,r) = \frac{(-1)^{k+l+r}}{k!l!r!}\Gamma\left(\frac{p_2K+\Delta''_{23}l-m_2R''}{\Delta''_{12}}\right)\Gamma\left(\frac{-p_1K+\Delta''_{13}l+m_1R''}{\Delta''_{12}}\right) \times\Gamma\left(\frac{\Delta''_{12}-\Delta'_{12}K-\Delta_{123}l-\Delta_{12}R''}{\nu\Delta''_{12}}\right)\lambda^{-\boldsymbol{\delta}\cdot\boldsymbol{t}},$$

with

$$-\boldsymbol{\delta}\cdot\boldsymbol{t} = \left(\frac{\Delta''_{12}}{\mu} + \frac{\Delta'_{12}}{\nu} + P_{12}\right)\frac{K}{\Delta''_{12}} + \left(\frac{\Delta_{123}}{\nu} + \Delta''_{12} - \Delta''_{13} - \Delta''_{23}\right)\frac{l}{\Delta''_{12}} + \left(\frac{\Delta_{12}}{\nu} + \frac{\Delta''_{12}}{\eta} - M_{12}\right)\frac{R''}{\Delta''_{12}}.$$

The primes appearing in the summation index r in (4.14) and (4.15) indicate that r is subject to the restrictions $t_1^{(2)} \leqslant 0$ and $t_1^{(3)} \leqslant 0$, respectively; recall (4.10) and (4.11).

The $t_2^{(2)}$ sequence of poles in (4.7) gives rise to the series of contributions (suppressing the leading factor of $\lambda^{-1/\mu-1/\nu-1/\eta}$)

$$I_{12} = \frac{1}{\eta \Delta_{23}} \sum_{k,l'} \frac{(-1)^{k+l}}{k!l!} \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \Gamma(t_1)\Gamma\left(\frac{-n_3K + m_3L' - \Delta_{13}t_1}{\Delta_{23}}\right) \times \Gamma\left(\frac{n_2K - m_2L' - \Delta_{12}t_1}{\Delta_{23}}\right) \times \Gamma\left(\frac{\Delta_{23} - \Delta'_{23}K - \Delta''_{23}L' + \Delta_{123}t_1}{\eta\Delta_{23}}\right) \lambda^{-\boldsymbol{\delta}\cdot\boldsymbol{t}}\,\mathrm{d}t_1, \tag{4.16}$$

where the prime attached to the summation index l indicates the presence of the constraint $t_2^{(2)} > 0$; recall (4.7). Upon setting $t_1 = \rho \mathrm{e}^{\mathrm{i}\theta}$, $|\theta| < \frac{1}{2}\pi$, in the integrands in this series, we find that the logarithm of the modulus of each integrand has the large-ρ behaviour

$$\rho\cos\theta\log\rho \cdot \left(\frac{\Delta_{123}}{\eta} - \Delta_{12} - \Delta_{13} + \Delta_{23}\right)/\Delta_{23} + \mathcal{O}(\rho). \tag{4.17}$$

The sign of the factor involving the various Δ can be deduced from the geometry of the Newton diagram.

Consider the plane through P_2, P_3 and $(0, 0, \eta)$. It has equation

$$\frac{\eta N_{23} - \Delta'_{23}}{\eta\Delta_{23}} m - \frac{\Delta''_{23} + \eta M_{23}}{\eta\Delta_{23}} n + \frac{p}{\eta} - 1 = 0,$$

and the evaluation of the left-hand side of this equation at the origin is clearly negative. Thus, the criterion

$$\frac{\eta N_{23} - \Delta'_{23}}{\eta\Delta_{23}} m - \frac{\Delta''_{23} + \eta M_{23}}{\eta\Delta_{23}} n + \frac{p}{\eta} - 1 < 0$$

is the condition satisfied by any point that lies in the half-space defined by this plane containing the origin.

If the Newton diagram is to be convex, then P_1 must not lie in this half-space, whence we must have

$$\frac{\eta N_{23} - \Delta'_{23}}{\eta\Delta_{23}} m_1 - \frac{\Delta''_{23} + \eta M_{23}}{\eta\Delta_{23}} n_1 + \frac{p_1}{\eta} - 1 > 0;$$

otherwise stated, the faces of the Newton diagram must 'bend' away from each other in such a fashion that the angles formed by adjacent faces meeting along a common edge be less than π, measured on the side of the Newton diagram not containing the origin. The left-hand side of this last inequality can be rewritten with the aid of the identities in the appendix to yield

$$\frac{1}{\Delta_{23}}\left(\frac{\Delta_{123}}{\eta} - \Delta_{12} - \Delta_{13} + \Delta_{23}\right) < 0.$$

We deduce, therefore, that the order estimate (4.17) must tend to $-\infty$ as $\rho \to \infty$, which in turn leads us to displace the t_1 integration contour in each integral of (4.16) to the left. In shifting these contours to the left, we encounter poles at

$$t_1^{(1)} = -r,$$

$$t_1^{(2)} = (-n_3K + m_3L' + \Delta_{23}r)/\Delta_{13} \leqslant 0, \tag{4.18}$$

$$t_1^{(3)} = (n_2K - m_2L' + \Delta_{23}r)/\Delta_{12} \leqslant 0, \tag{4.19}$$

$$t_1^{(4)} = (\Delta'_{23}K + \Delta''_{23}L' - \Delta_{23}R'')/\Delta_{123} \leqslant 0, \tag{4.20}$$

where r is once again a non-negative integer and the K, L and R are defined in (2.7), (3.12) and (4.12), respectively.

The $t_1^{(1)}$ sequence gives rise to the asymptotic series (without the usual leading power of λ)

$$I_{121} = \frac{1}{\eta\Delta_{23}} \sum_{k,l',r} f_{121}(k,l,r), \tag{4.21}$$

where

$$f_{121}(k,l,r) = \frac{(-1)^{k+l+r}}{k!l!r!} \Gamma\left(\frac{-n_3K + m_3L' + \Delta_{13}r}{\Delta_{23}}\right) \Gamma\left(\frac{n_2K - m_2L' + \Delta_{12}r}{\Delta_{23}}\right) \times \Gamma\left(\frac{\Delta_{23} - \Delta'_{23}K - \Delta''_{23}L' - \Delta_{123}r}{\eta\Delta_{23}}\right) \lambda^{-\boldsymbol{\delta}\cdot\boldsymbol{t}},$$

with

$$-\boldsymbol{\delta}\cdot\boldsymbol{t} = \left(\frac{\Delta_{23}}{\mu} + \frac{\Delta'_{23}}{\eta} - N_{23}\right)\frac{K}{\Delta_{23}} + \left(\frac{\Delta_{23}}{\nu} + \frac{\Delta''_{23}}{\eta} + M_{23}\right)\frac{L'}{\Delta_{23}} + \left(\frac{\Delta_{123}}{\eta} - \Delta_{12} - \Delta_{13} + \Delta_{23}\right)\frac{r}{\Delta_{23}}.$$

The sequence of poles $t_1^{(2)}$ yields the series

$$I_{122} = -\frac{1}{\eta\Delta_{13}} \sum_{k,l',r'} f_{122}(k,l,r), \tag{4.22}$$

where

$$f_{122}(k,l,r) = \frac{(-1)^{k+l+r}}{k!l!r!} \Gamma\left(\frac{-n_3K + m_3L' + \Delta_{23}r}{\Delta_{13}}\right) \Gamma\left(\frac{n_1K - m_1L' - \Delta_{12}r}{\Delta_{13}}\right) \times \Gamma\left(\frac{\Delta_{13} - \Delta'_{13}K + \Delta''_{13}L' + \Delta_{123}r}{\eta\Delta_{13}}\right) \lambda^{-\boldsymbol{\delta}\cdot\boldsymbol{t}},$$

with

$$-\boldsymbol{\delta}\cdot\boldsymbol{t} = \left(\frac{\Delta_{13}}{\mu} + \frac{\Delta'_{13}}{\eta} - N_{13}\right)\frac{K}{\Delta_{13}} + \left(\frac{\Delta_{13}}{\nu} - \frac{\Delta''_{13}}{\eta} + M_{13}\right)\frac{L'}{\Delta_{13}} + \left(-\frac{\Delta_{123}}{\eta} + \Delta_{12} + \Delta_{13} - \Delta_{23}\right)\frac{r}{\Delta_{13}}.$$

The $t_1^{(3)}$ sequence gives rise to the series

$$I_{123} = -\frac{1}{\eta\Delta_{12}} \sum_{k,l',r'} f_{112}(k,r,l), \tag{4.23}$$

while, finally, the $t_1^{(4)}$ sequence leads to the series

$$I_{124} = \frac{1}{\Delta_{123}} \sum_{k,l',r'} f_{124}(k,l,r), \tag{4.24}$$

where

$$\begin{aligned} f_{124}(k,l,r) = \frac{(-1)^{k+l+r}}{k!l!r!} &\Gamma\left(\frac{\Delta'_{23}K + \Delta''_{23}L' - \Delta_{23}R''}{\Delta_{123}}\right) \\ &\times \Gamma\left(\frac{-\Delta'_{13}K + \Delta''_{13}L' + \Delta_{13}R''}{\Delta_{123}}\right) \\ &\times \Gamma\left(\frac{\Delta'_{12}K - \Delta''_{12}L' + \Delta_{12}R''}{\Delta_{123}}\right) \lambda^{-\boldsymbol{\delta}\cdot\boldsymbol{t}}, \end{aligned} \tag{4.25}$$

with

$$\begin{aligned} -\boldsymbol{\delta}\cdot\boldsymbol{t} = \left(\frac{\Delta_{123}}{\mu} - \Delta'_{12} + \Delta'_{13} - \Delta'_{23}\right)\frac{K}{\Delta_{123}} + \left(\frac{\Delta_{123}}{\nu} + \Delta''_{12} - \Delta''_{13} - \Delta''_{23}\right)\frac{L'}{\Delta_{123}} \\ + \left(\frac{\Delta_{123}}{\eta} - \Delta_{12} - \Delta_{13} + \Delta_{23}\right)\frac{R''}{\Delta_{123}}. \end{aligned}$$

As before, the primes appearing on summation indices signal the presence of restrictions. In the case of l', common to (4.21), (4.22), (4.23) and (4.24), this is the inequality that results from setting $t_1 = 0$ in (4.7). For the restriction on r in each of (4.22), (4.23) and (4.24), this arises from (4.18), (4.19) and (4.20), respectively.

We turn now to the $t_2^{(3)}$ sequence of poles, given in (4.8). These poles give rise to the series of contributions (suppressing, as usual, the leading power of λ)

$$\begin{aligned} I_{13} = \frac{1}{\nu\Delta''_{23}} \sum_{k,l'} \frac{(-1)^{k+l}}{k!l!} \frac{1}{2\pi\mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} &\Gamma(t_1)\Gamma\left(\frac{-p_3K + m_3L'' + \Delta''_{13}t_1}{\Delta''_{23}}\right) \\ \times &\Gamma\left(\frac{p_2K - m_2L'' - \Delta''_{12}t_1}{\Delta''_{23}}\right) \\ \times &\Gamma\left(\frac{\Delta''_{23} + \Delta'_{23}K - \Delta_{23}L'' - \Delta_{123}t_1}{\nu\Delta''_{23}}\right)\lambda^{-\boldsymbol{\delta}\cdot\boldsymbol{t}}\,\mathrm{d}t_1, \end{aligned} \tag{4.26}$$

where the prime attached to the summation index l indicates that l is subject to the restriction (4.8), with $t_1 = 0$. Setting $t_1 = \rho\mathrm{e}^{\mathrm{i}\theta}$, $|\theta| < \frac{1}{2}\pi$, we arrive at the large-ρ estimate of the behaviour of the logarithms of the moduli of the integrands given by

$$\rho\cos\theta\log\rho\cdot\left(-\frac{\Delta_{123}}{\nu} - \Delta''_{12} + \Delta''_{13} + \Delta''_{23}\right)/\Delta''_{23} + \mathcal{O}(\rho). \tag{4.27}$$

As before, we can deduce the sign of this quantity by considering the shape of the Newton diagram.

The plane containing the vertices $(0,\nu,0)$, P_1 and P_2 has the equation

$$\frac{(-\Delta'_{12} - \nu\Delta_{12})}{\nu\Delta''_{12}}m + \frac{n}{\nu} + \frac{(-\Delta_{12} + \nu M_{12})}{\nu\Delta''_{12}}p - 1 = 0.$$

Since P_3 must lie in the half-space determined by this plane, not containing the origin, convexity of the Newton diagram requires

$$\frac{(-\Delta'_{12} - \nu\Delta_{12})}{\nu\Delta''_{12}}m_3 + \frac{n_3}{\nu} + \frac{(-\Delta_{12} + \nu M_{12})}{\nu\Delta''_{12}}p_3 - 1 > 0.$$

The left-hand side of this inequality can be recast with the aid of the first identity in (A 7) to arrive at

$$\frac{1}{\Delta''_{12}}\left(\frac{\Delta_{123}}{\nu}+\Delta''_{12}-\Delta''_{13}-\Delta''_{23}\right)<0. \tag{4.28}$$

This, however, is not yet sufficient to determine the sign of (4.27).

Observe that $\overleftrightarrow{P_1P_2}$ meets the np-plane at $(0, \Delta_{12}/M_{12}, \Delta''_{12}/M_{12})$. In our configuration (recall (4.2)), this occurs for $\Delta_{12}/M_{12}>0$ and $\Delta''_{12}/M_{12}>0$, and since $M_{12}>0$ we must have $\Delta''_{12}>0$. Therefore, we can reduce (4.28) to

$$\frac{\Delta_{123}}{\nu}+\Delta''_{12}-\Delta''_{13}-\Delta''_{23}<0. \tag{4.29}$$

It remains to determine the sign of Δ''_{23} to determine the sign of (4.27).

The line $\overleftrightarrow{P_2P_3}$ meets the mn-plane at $(\Delta''_{23}/P_{23}, -\Delta'_{23}/P_{23}, 0)$ with $\Delta''_{23}/P_{23}>0$ and $-\Delta'_{23}/P_{23}<0$. The assumption $P_{23}>0$ (recall (4.2)) then implies that both Δ''_{23} and Δ'_{23} are positive. With this and (4.29), we conclude that the factor

$$\frac{1}{\Delta''_{23}}\left(-\frac{\Delta_{123}}{\nu}-\Delta''_{12}+\Delta''_{13}+\Delta''_{23}\right)<0,$$

so that (4.27) tends to $+\infty$ as $\rho\to\infty$, and so the t_1 integration contours appearing in (4.26) must be shifted to the right to extract the asymptotic behaviour.

In displacing the t_1 contours to the right, we encounter contributions arising from three sequences of poles:

$$t_1^{(1)}=(p_3K-m_3L''-\Delta''_{23}r)/\Delta''_{13}>0, \tag{4.30}$$

$$t_1^{(2)}=(p_2K-m_2L''+\Delta''_{23}r)/\Delta''_{12}>0, \tag{4.31}$$

$$t_1^{(3)}=(\Delta'_{23}K-\Delta_{23}L''+\Delta''_{23}R')/\Delta_{123}>0, \tag{4.32}$$

where r is a non-negative integer, and the K, L and R are given, respectively, in (2.7), (3.12) and (4.12).

The $t_1^{(1)}$ sequence of poles yields the asymptotic series

$$I_{131}=-\frac{1}{\nu\Delta''_{13}}\sum_{k,l',r'}f_{131}(k,l,r), \tag{4.33}$$

where

$$f_{131}(k,l,r)=\frac{(-1)^{k+l+r}}{k!l!r!}\Gamma\left(\frac{p_3K-m_3L''-\Delta''_{23}r}{\Delta''_{13}}\right)\Gamma\left(\frac{-p_1K+m_1L''+\Delta''_{12}r}{\Delta''_{13}}\right)$$
$$\times\Gamma\left(\frac{\Delta''_{13}-\Delta'_{13}K+\Delta_{13}L''+\Delta_{123}r}{\nu\Delta''_{13}}\right)\lambda^{-\boldsymbol{\delta}\cdot\boldsymbol{t}},$$

with

$$-\boldsymbol{\delta}\cdot\boldsymbol{t}=\left(\frac{\Delta''_{13}}{\mu}+\frac{\Delta'_{13}}{\nu}+P_{13}\right)\frac{K}{\Delta''_{13}}+\left(-\frac{\Delta_{13}}{\nu}+\frac{\Delta''_{13}}{\eta}-M_{13}\right)\frac{L''}{\Delta''_{13}}$$
$$+\left(-\frac{\Delta_{123}}{\nu}-\Delta''_{12}+\Delta''_{13}+\Delta''_{23}\right)\frac{r}{\Delta''_{13}}.$$

The $t_1^{(2)}$ and $t_1^{(3)}$ sequences contribute, respectively,

$$I_{132}=\frac{1}{\nu\Delta''_{12}}\sum_{k,l',r'}f_{113}(k,r,l), \tag{4.34}$$

$$I_{133} = \frac{1}{\Delta_{123}} \sum_{k,l',r'} f_{124}(k,r,l). \tag{4.35}$$

As we have seen before, primes associated with summation indices indicate restrictions on the ranges of the indices. In each of (4.33), (4.34) and (4.35), the r index is restricted by (4.30)–(4.32), respectively. In addition, all three of these series are subject to the l restriction obtained from (4.8) by setting $t_1 = 0$.

(*b*) *Treatment of the series of integrals* I_2

For the I_2 series in (4.4), we displace t_2 integration contours to the left, so poles we encounter arise in the following sequences:

$$t_2^{(1)} = -l,$$

$$t_2^{(2)} = (m_3K' - n_3L - \Delta_{13}t_1)/\Delta_{23} \leqslant 0, \tag{4.36}$$

$$t_2^{(3)} = (p_3K' - n_3L'' - \Delta'_{13}t_1)/\Delta'_{23} \leqslant 0; \tag{4.37}$$

here, l is a non-negative integer.

The $t_2^{(1)}$ sequence gives rise to the series (suppressing, as usual, the leading power of λ)

$$I_{21} = \frac{1}{\mu n_3 \eta} \sum_{k,l} \frac{(-1)^{k+l}}{k!l!} \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \Gamma(t_1)\Gamma\left(\frac{K' + n_2 l - n_1 t_1}{n_3}\right)$$
$$\times \Gamma\left(\frac{n_3 - m_3K' - \Delta_{23}l + \Delta_{13}t_1}{\mu n_3}\right)$$
$$\times \Gamma\left(\frac{n_3 - p_3K' - \Delta'_{23}l + \Delta'_{13}t_1}{n_3\eta}\right)\lambda^{-\boldsymbol{\delta}\cdot\boldsymbol{t}}\,\mathrm{d}t_1.$$

Putting $t_1 = \rho \mathrm{e}^{\mathrm{i}\theta}$, $|\theta| < \frac{1}{2}\pi$, in the integrands of I_{21}, we find these integrands have moduli whose logarithms have the large-ρ behaviour

$$\rho\cos\theta\log\rho \cdot \left(\frac{\Delta_{13}}{\mu} + \frac{\Delta'_{13}}{\eta} - N_{13}\right)/n_3 + \mathcal{O}(\rho),$$

and since $\overleftrightarrow{P_1P_3}$ meets the mp-plane at $(\Delta_{13}/N_{13}, 0, \Delta'_{13}/N_{13})$, the convexity requirement

$$\frac{(\Delta_{13}/N_{13})}{\mu} + \frac{(\Delta'_{13}/N_{13})}{\eta} - 1 < 0,$$

allows us to deduce that the order estimate for the integrands tends to $-\infty$ as $\rho \to \infty$; recall the discussion leading up to (3.8). Accordingly, the t_1 integration contours in I_{21} must be displaced to the left, encountering the following sequences of poles:

$$t_1^{(1)} = -r,$$

$$t_1^{(2)} = (m_3K' + \Delta_{23}l - n_3R)/\Delta_{13} \leqslant 0, \tag{4.38}$$

$$t_1^{(3)} = (p_3K' + \Delta'_{23}l - n_3R'')/\Delta'_{13} \leqslant 0; \tag{4.39}$$

here, r is a non-negative integer.

The $t_1^{(1)}$ sequence of poles generates the series

$$I_{\mu n_3 \eta} \equiv I_{211} = \frac{1}{\mu n_3 \eta} \sum_{k,l,r} f_{211}(k,l,r), \tag{4.40}$$

MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES
PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A

where

$$f_{211}(k,l,r) = \frac{(-1)^{k+l+r}}{k!l!r!}\Gamma\left(\frac{K'+n_2l+n_1r}{n_3}\right)\Gamma\left(\frac{n_3-m_3K'-\Delta_{23}l-\Delta_{13}r}{\mu n_3}\right) \times\Gamma\left(\frac{n_3-p_3K'-\Delta'_{23}l-\Delta'_{13}r}{n_3\eta}\right)\lambda^{-\boldsymbol{\delta}\cdot\boldsymbol{t}},$$

with

$$-\boldsymbol{\delta}\cdot\boldsymbol{t} = -\frac{\delta_3}{n_3}K' + \left(\frac{\Delta_{23}}{\mu}+\frac{\Delta'_{23}}{\eta}-N_{23}\right)\frac{l}{n_3} + \left(\frac{\Delta_{13}}{\mu}+\frac{\Delta'_{13}}{\eta}-N_{13}\right)\frac{r}{n_3},$$

while the $t_1^{(2)}$ and $t_1^{(3)}$ sequences yield

$$I_{212} = \frac{1}{\eta\Delta_{13}}\sum_{k,l,r'} f_{122}(r,k,l), \tag{4.41}$$

$$I_{213} = \frac{1}{\mu\Delta'_{13}}\sum_{k,l,r'} f_{213}(k,l,r), \tag{4.42}$$

where

$$f_{213}(k,l,r) = \frac{(-1)^{k+l+r}}{k!l!r!}\Gamma\left(\frac{p_3K'+\Delta'_{23}l-n_3R''}{\Delta'_{13}}\right)\Gamma\left(\frac{-p_1K'+\Delta'_{12}l+n_1R''}{\Delta'_{13}}\right) \times\Gamma\left(\frac{\Delta'_{13}-\Delta''_{13}K'-\Delta_{123}l-\Delta_{13}R''}{\mu\Delta'_{13}}\right)\lambda^{-\boldsymbol{\delta}\cdot\boldsymbol{t}},$$

with

$$-\boldsymbol{\delta}\cdot\boldsymbol{t} = \left(\frac{\Delta''_{13}}{\mu}+\frac{\Delta'_{13}}{\nu}+P_{13}\right)\frac{K'}{\Delta'_{13}} + \left(\frac{\Delta_{123}}{\mu}-\Delta'_{12}+\Delta'_{13}-\Delta'_{23}\right)\frac{l}{\Delta'_{13}} + \left(\frac{\Delta_{13}}{\mu}+\frac{\Delta'_{13}}{\eta}-N_{13}\right)\frac{R''}{\Delta'_{13}}.$$

The primes appearing in the sums (4.41) and (4.42) indicate the presence of the restrictions (4.38) and (4.39).

The $t_2^{(2)}$ sequence in (4.36) gives rise to the series (suppressing, as usual, the leading power of λ)

$$I_{22} = \frac{1}{\eta\Delta_{23}}\sum_{k,l'}\frac{(-1)^{k+l}}{k!l!}\frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\Gamma(t_1)\Gamma\left(\frac{m_3K'-n_3L-\Delta_{13}t_1}{\Delta_{23}}\right) \times\Gamma\left(\frac{-m_2K'+n_2L-\Delta_{12}t_1}{\Delta_{23}}\right)\Gamma\left(\frac{\Delta_{23}-\Delta''_{23}K'-\Delta'_{23}L+\Delta_{123}t_1}{\eta\Delta_{23}}\right)\lambda^{-\boldsymbol{\delta}\cdot\boldsymbol{t}}\,\mathrm{d}t_1.$$

Putting $t_1 = \rho\mathrm{e}^{\mathrm{i}\theta}$, $|\theta| < \frac{1}{2}\pi$, in the integrands of I_{22}, we find the logarithms of the moduli of the integrands have large-ρ behaviour

$$\rho\cos\theta\log\rho\cdot\left(\frac{\Delta_{123}}{\eta}-\Delta_{12}-\Delta_{13}+\Delta_{23}\right)/\Delta_{23}+\mathcal{O}(\rho).$$

The convexity argument employed for I_{12} applied here allows us to deduce that this order estimate tends to $-\infty$ as $\rho\to\infty$. Accordingly, the t_1 integration contours in

I_{22} must be displaced to the left, encountering the following sequences of poles:

$$t_1^{(1)} = -r,$$
$$t_1^{(2)} = (m_3K' - n_3L + \Delta_{23}r)/\Delta_{13} \leqslant 0, \tag{4.43}$$
$$t_1^{(3)} = (-m_2K' + n_2L + \Delta_{23}r)/\Delta_{12} \leqslant 0, \tag{4.44}$$
$$t_1^{(4)} = (\Delta''_{23}K' + \Delta'_{23}L - \Delta_{23}R'')/\Delta_{123} \leqslant 0; \tag{4.45}$$

as usual, r is a non-negative integer. These sequences yield the asymptotic series

$$I_{221} = \frac{1}{\eta\Delta_{23}}\sum_{k,l',r} f_{121}(l,k,r),$$
$$I_{222} = -\frac{1}{\eta\Delta_{13}}\sum_{k,l',r'} f_{122}(l,k,r),$$
$$I_{223} = -\frac{1}{\eta\Delta_{12}}\sum_{k,l',r'} f_{112}(l,r,k),$$
$$I_{224} = \frac{1}{\Delta_{123}}\sum_{k,l',r'} f_{124}(l,k,r), \tag{4.46}$$

respectively. The primes appearing on the summation indices r indicate the restrictions resulting from (4.43)–(4.45), and the prime associated with the indices l mark the restriction obtained from (4.36) obtained by setting $t_1 = 0$.

The sequence of poles $t_2^{(3)}$ from (4.37) yields the formal series (again suppressing the leading power of λ)

$$I_{23} = \frac{1}{\mu\Delta'_{23}}\sum_{k,l'}\frac{(-1)^{k+l}}{k!l!}\frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\Gamma(t_1)\Gamma\left(\frac{p_3K' - n_3L'' - \Delta'_{13}t_1}{\Delta'_{23}}\right)$$
$$\times\Gamma\left(\frac{-p_2K' + n_2L'' + \Delta'_{12}t_1}{\Delta'_{23}}\right)\Gamma\left(\frac{\Delta'_{23} + \Delta''_{23}K' - \Delta_{23}L'' - \Delta_{123}t_1}{\mu\Delta'_{23}}\right)\lambda^{-\boldsymbol{\delta}\cdot\boldsymbol{t}}\,\mathrm{d}t_1.$$

Putting $t_1 = \rho\mathrm{e}^{\mathrm{i}\theta}$, $|\theta| < \frac{1}{2}\pi$, in the integrands of I_{22}, we find that the logarithm of the modulus of each integrand has the the large-ρ behaviour

$$\rho\cos\theta\log\rho\cdot\left(-\frac{\Delta_{123}}{\mu} + \Delta'_{12} - \Delta'_{13} + \Delta'_{23}\right)/\Delta'_{23} + \mathcal{O}(\rho). \tag{4.47}$$

We again employ a geometric argument to deduce the sign of the last factor.

The plane generated by $(\mu, 0, 0)$, P_1 and P_3 has equation

$$\frac{m}{\mu} - \frac{(\Delta''_{13} + \mu P_{13})}{\mu\Delta'_{13}}n + \frac{(\mu N_{13} - \Delta_{13})}{\mu\Delta'_{13}}p - 1 = 0,$$

and by virtue of the convexity of the Newton diagram, P_2 must lie in the half-space determined by this plane, not containing the origin, i.e. P_2 must satisfy

$$\frac{m_2}{\mu} - \frac{(\Delta''_{13} + \mu P_{13})}{\mu\Delta'_{13}}n_2 + \frac{(\mu N_{13} - \Delta_{13})}{\mu\Delta'_{13}}p_2 - 1 > 0.$$

Rewriting the left-hand side of this inequality with the aid of the fifth identity of

(A 7), we find this is equivalent to

$$\frac{1}{\Delta'_{13}}(\frac{\Delta_{123}}{\mu} - \Delta'_{12} + \Delta'_{13} - \Delta'_{23}) < 0,$$

which, in view of the fact that $\Delta'_{13} > 0$ (recall (4.1)), yields

$$\frac{\Delta_{123}}{\mu} - \Delta'_{12} + \Delta'_{13} - \Delta'_{23} < 0.$$

The reasoning following (4.29) showed that $\Delta'_{23} > 0$, whence it follows that

$$\frac{1}{\Delta'_{23}}(\frac{\Delta_{123}}{\mu} - \Delta'_{12} + \Delta'_{13} - \Delta'_{23}) < 0.$$

With this in hand, we see that the order estimate (4.47) must tend to $+\infty$ as $\rho \to \infty$, and so we must displace the integration contours in I_{23} to the right to determine its asymptotic behaviour. As we shift the t_1 contours, we encounter three sequences of poles, namely,

$$t_1^{(1)} = (p_3 K' - n_3 L'' + \Delta'_{23} r)/\Delta'_{13} > 0, \tag{4.48}$$

$$t_1^{(2)} = (p_2 K' - n_2 L'' - \Delta'_{23} r)/\Delta'_{12} > 0, \tag{4.49}$$

$$t_1^{(3)} = (\Delta''_{23} K' - \Delta_{23} L'' + \Delta'_{23} R)/\Delta_{123} > 0. \tag{4.50}$$

The $t_1^{(1)}$ and $t_1^{(3)}$ sequences produce the formal series

$$\begin{aligned} I_{231} &= \frac{1}{\mu\Delta'_{13}} \sum_{k,l',r'} f_{213}(k,r,l), \\ I_{233} &= \frac{1}{\Delta_{123}} \sum_{k,l',r'} f_{124}(r,k,l), \end{aligned} \tag{4.51}$$

while that of $t_1^{(2)}$ yields

$$I_{232} = -\frac{1}{\mu\Delta'_{12}} \sum_{k,l',r'} f_{232}(k,l,r),$$

where

$$\begin{aligned} f_{232}(k,l,r) = \frac{(-1)^{k+l+r}}{k!l!r!} \Gamma\left(\frac{p_2 K' - n_2 L'' - \Delta'_{23} r}{\Delta'_{12}}\right) \Gamma\left(\frac{-p_1 K' + n_1 L'' + \Delta'_{13} r}{\Delta'_{12}}\right) \\ \times \Gamma\left(\frac{\Delta'_{12} - \Delta''_{12} K' + \Delta_{12} L'' + \Delta_{123} r}{\mu\Delta'_{12}}\right) \lambda^{-\boldsymbol{\delta}\cdot\boldsymbol{t}}, \end{aligned}$$

with

$$\begin{aligned} -\boldsymbol{\delta}\cdot\boldsymbol{t} = \left(\frac{\Delta''_{12}}{\mu} + \frac{\Delta'_{12}}{\nu} + P_{12}\right)\frac{K'}{\Delta'_{12}} + \left(-\frac{\Delta_{12}}{\mu} + \frac{\Delta'_{12}}{\eta} - N_{12}\right)\frac{L''}{\Delta'_{12}} \\ + \left(-\frac{\Delta_{123}}{\mu} + \Delta'_{12} - \Delta'_{13} + \Delta'_{23}\right)\frac{r}{\Delta'_{12}}. \end{aligned}$$

As usual, the primes appearing with the summation indices indicate restrictions: those for r arise from (4.48)–(4.50), and those for l from (4.37) with $t_1 = 0$.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

(*c*) *Treatment of the series of integrals I_3*

Finally, we address the analysis of I_3 given in (4.5), for which the t_2 integration contours are to be displaced to the left. Poles of the integrand that are encountered as we shift contours form the three sequences

$$t_2^{(1)} = -l,$$
$$t_2^{(2)} = (-n_3 K'' + p_3 L' - \Delta'_{13} t_1)/\Delta'_{23} \leqslant 0, \tag{4.52}$$
$$t_2^{(3)} = (m_3 K'' - p_3 L + \Delta''_{13} t_1)/\Delta''_{23} \leqslant 0, \tag{4.53}$$

with l a non-negative integer. The first of these gives rise to the series of contributions (suppressing a leading factor of $\lambda^{-1/\mu-1/\nu-1/\eta}$)

$$I_{31} = \frac{1}{\mu\nu p_3} \sum_{k,l} \frac{(-1)^{k+l}}{k!l!} \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \Gamma(t_1) \Gamma\left(\frac{K'' + p_2 l - p_1 t_1}{p_3}\right) \times \Gamma\left(\frac{p_3 - m_3 K'' - \Delta''_{23} l - \Delta''_{13} t_1}{\mu p_3}\right) \times \Gamma\left(\frac{p_3 - n_3 K'' + \Delta'_{23} l - \Delta'_{13} t_1}{\nu p_3}\right) \lambda^{-\boldsymbol{\delta}\cdot\boldsymbol{t}}\, \mathrm{d}t_1. \tag{4.54}$$

Upon setting $t_1 = \rho \mathrm{e}^{\mathrm{i}\theta}$, $|\theta| < \frac{1}{2}\pi$, in the integrands of I_{31}, we find the moduli of the integrands have logarithms with the large-ρ behaviour

$$\rho \cos\theta \log\rho \cdot \left(-\frac{\Delta''_{13}}{\mu} - \frac{\Delta'_{13}}{\nu} - P_{13}\right) / p_3 + \mathcal{O}(\rho).$$

Since $\overleftrightarrow{P_1 P_3}$ meets the mn-plane at $(-\Delta''_{13}/P_{13}, -\Delta'_{13}/P_{13}, 0)$ and convexity of the Newton diagram requires that

$$\frac{(-\Delta''_{13}/P_{13})}{\mu} + \frac{(-\Delta'_{13}/P_{13})}{\nu} - 1 < 0,$$

use of the fact $P_{13} < 0$ (recall (4.2)) allows us to deduce that the order estimate above tends to $+\infty$ as $\rho \to \infty$. Thus, we shift the t_1 contours in (4.54) to the right to obtain large-λ asymptotic behaviour and encounter three sequences of poles in the process:

$$t_1^{(1)} = (K'' + p_2 l + p_3 r)/p_1,$$
$$t_1^{(2)} = (-m_3 K'' - \Delta''_{23} l + p_3 R)/\Delta''_{13} > 0, \tag{4.55}$$
$$t_1^{(3)} = (-n_3 K'' + \Delta'_{23} l + p_3 R')/\Delta'_{13} > 0, \tag{4.56}$$

here, as before, r is a non-negative integer.

The $t_1^{(1)}$ sequence gives rise to the series of contributions

$$I_{\mu\nu p_1} \equiv I_{311} = \frac{1}{\mu\nu p_1} \sum_{k,l,r} f_{311}(k,l,r), \tag{4.57}$$

where

$$f_{311}(k,l,r) = \frac{(-1)^{k+l+r}}{k!l!r!} \Gamma\left(\frac{K'' + p_2 l + p_3 r}{p_1}\right) \Gamma\left(\frac{p_1 - m_1 K'' - \Delta''_{12} l - \Delta''_{13} r}{\mu p_1}\right) \times \Gamma\left(\frac{p_1 - n_1 K'' - \Delta'_{12} l - \Delta'_{13} r}{\nu p_1}\right) \lambda^{-\boldsymbol{\delta}\cdot\boldsymbol{t}},$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

with

$$-\boldsymbol{\delta}\cdot\boldsymbol{t} = -\frac{\delta_1}{p_1}K'' + \left(\frac{\Delta''_{12}}{\mu} + \frac{\Delta'_{12}}{\nu} + P_{12}\right)\frac{l}{p_1} + \left(\frac{\Delta''_{13}}{\mu} + \frac{\Delta'_{13}}{\nu} + P_{13}\right)\frac{r}{p_1},$$

while the $t_1^{(2)}$ and $t_1^{(3)}$ sequences yield

$$I_{312} = \frac{1}{\nu\Delta''_{13}}\sum_{k,l,r'} f_{131}(r,k,l),$$

$$I_{313} = \frac{1}{\mu\Delta'_{13}}\sum_{k,l,r'} f_{213}(r,l,k).$$

The primes associated with the summation index r in each of the latter two series arise from the constraints imposed in (4.55) and (4.56).

The $t_2^{(2)}$ sequence in (4.52) leads to the series (suppressing the leading power of λ)

$$I_{32} = -\frac{1}{\mu\Delta'_{23}}\sum_{k,l'}\frac{(-1)^{k+l}}{k!l!}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\Gamma(t_1)\Gamma\left(\frac{-n_3K''+p_3L'-\Delta'_{13}t_1}{\Delta'_{23}}\right)$$
$$\times\Gamma\left(\frac{n_2K''-p_2L'+\Delta'_{12}t_1}{\Delta'_{23}}\right)$$
$$\times\Gamma\left(\frac{\Delta'_{23}-\Delta_{23}K''+\Delta''_{23}L'-\Delta_{123}t_1}{\mu\Delta'_{23}}\right)\lambda^{-\boldsymbol{\delta}\cdot\boldsymbol{t}}\,dt_1.$$

With $t_1=\rho e^{i\theta}$, $|\theta|<\frac{1}{2}\pi$, in the integrands of I_{32}, we find the integrands have large-ρ behaviour

$$\rho\cos\theta\log\rho\cdot\left(-\frac{\Delta_{123}}{\mu}+\Delta'_{12}-\Delta'_{13}+\Delta'_{23}\right)/\Delta'_{23}+\mathcal{O}(\rho).$$

Reasoning as we did for the order estimate for I_{23} (following (4.47)), we conclude that this order estimate tends to $+\infty$ as $\rho\to\infty$, and so displace integration contours to the right. Poles of the integrands that we encounter in shifting these contours appear in three sequences,

$$t_1^{(1)} = (-n_3K''+p_3L'+\Delta'_{23}r)/\Delta'_{13} > 0, \tag{4.58}$$

$$t_1^{(2)} = (-n_2K''+p_2L'-\Delta'_{23}r)/\Delta'_{12} > 0, \tag{4.59}$$

$$t_1^{(3)} = (-\Delta_{23}K''+\Delta''_{23}L'+\Delta'_{23}R)/\Delta_{123} > 0, \tag{4.60}$$

where r is a non-negative integer. In turn, each of these sequences of poles gives rise to the formal asymptotic series

$$\left.\begin{aligned} I_{321} &= -\frac{1}{\mu\Delta'_{13}}\sum_{k,l',r'} f_{213}(l,r,k),\\ I_{322} &= \frac{1}{\mu\Delta'_{12}}\sum_{k,l',r'} f_{232}(l,k,r),\\ I_{323} &= -\frac{1}{\Delta_{123}}\sum_{k,l',r'} f_{124}(r,l,k), \end{aligned}\right\} \tag{4.61}$$

where the prime associated with each summation index l arises from the constraint imposed by (4.52) with $t_1=0$, and the restrictions on indices r arise from (4.58)–(4.60).

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

To complete the asymptotic analysis of I_3, we turn to the $t_2^{(3)}$ sequence of poles (4.53). This sequence yields the formal series

$$I_{33} = \frac{1}{\nu \Delta''_{23}} \sum_{k,l'} \frac{(-1)^{k+l}}{k!l!} \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \Gamma(t_1) \Gamma\left(\frac{m_3 K'' - p_3 L + \Delta''_{13} t_1}{\Delta''_{23}}\right) \times \Gamma\left(\frac{-m_2 K'' + p_2 L - \Delta''_{12} t_1}{\Delta''_{23}}\right) \times \Gamma\left(\frac{\Delta''_{23} - \Delta_{23} K'' + \Delta'_{23} L - \Delta_{123} t_1}{\nu \Delta''_{23}}\right) \lambda^{-\boldsymbol{\delta}\cdot\boldsymbol{t}}\, \mathrm{d}t_1.$$

With $t_1 = \rho \mathrm{e}^{\mathrm{i}\theta}$, $|\theta| < \frac{1}{2}\pi$, in the integrands of I_{32}, we find the integrands have moduli with logarithms exhibiting the large-ρ behaviour

$$\rho \cos\theta \log\rho \cdot \left(-\frac{\Delta_{123}}{\nu} - \Delta''_{12} + \Delta''_{13} + \Delta''_{23}\right) / \Delta''_{23} + \mathcal{O}(\rho).$$

Arguing as we did for I_{13} (recall the discussion following (4.26)), we deduce that this estimate tends to $+\infty$ as $\rho \to \infty$, so that the t_1 integration contours must be translated to the right to develop the large-λ asymptotics of each integral. Again, three sequences of poles are encountered in the displacement of the contours,

$$t_1^{(1)} = (-m_3 K'' + p_3 L - \Delta''_{23} r)/\Delta''_{13} > 0, \tag{4.62}$$

$$t_1^{(2)} = (-m_2 K'' + p_2 L + \Delta''_{23} r)/\Delta''_{12} > 0, \tag{4.63}$$

$$t_1^{(3)} = (-\Delta_{23} K'' + \Delta'_{23} L + \Delta''_{23} R')/\Delta_{123} > 0, \tag{4.64}$$

with r a non-negative integer. These sequences of poles yield the formal series

$$\left.\begin{aligned} I_{331} &= -\frac{1}{\nu \Delta''_{13}} \sum_{k,l',r'} f_{131}(l,k,r), \\ I_{332} &= \frac{1}{\nu \Delta''_{12}} \sum_{k,l',r'} f_{113}(l,r,k), \\ I_{333} &= \frac{1}{\Delta_{123}} \sum_{k,l',r'} f_{124}(l,r,k), \end{aligned}\right\} \tag{4.65}$$

respectively, with the restrictions on l arising from (4.53) with $t_1 = 0$ and the restriction on each r determined by the restriction on the corresponding sequence of poles in (4.62)–(4.64).

This completes the evaluation of all contributions to the asymptotic expansion of $I(\lambda)$. As we saw in (I) and in § 3 of the present work, substantial simplification of the sum of all our formal series I_{ijk} is possible. Because of the already lengthy treatment in this case, we shall only provide details for one such simplification; the rest can be handled in similar fashion by the reader.

(*d*) *Final form for the asymptotic expansion*

Let us turn to all those series I_{ijk} which involve a leading factor of $\pm 1/\Delta_{123}$, namely I_{124}, I_{133}, I_{224}, I_{233}, I_{323} and I_{333}; cf. (4.24), (4.35), (4.46), (4.51), (4.61) and (4.65), respectively. Each of these six series has its summation indices governed by a pair of inequalities, restrictions that arose from requiring $t_2^{(\cdot)}$ and $t_1^{(\cdot)}$ to be either

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Table 1. *Inequalities governing summation indices for series with leading factor* $\pm 1/\Delta_{123}$

series	inequality from $t_2^{(\cdot)}$	inequality from $t_1^{(\cdot)}$
I_{124}	$(-n_3K + m_3L')/\Delta_{23} > 0$	$(\Delta'_{23}K + \Delta''_{23}L' - \Delta_{23}R'')/\Delta_{123} \leqslant 0$
I_{133}	$(-p_3K + m_3L'')/\Delta''_{23} > 0$	$(\Delta'_{23}K - \Delta_{23}L'' + \Delta''_{23}R')/\Delta_{123} > 0$
I_{224}	$(m_3K' - n_3L)/\Delta_{23} \leqslant 0$	$(\Delta''_{23}K' + \Delta'_{23}L - \Delta_{23}R'')/\Delta_{123} \leqslant 0$
I_{233}	$(p_3K' - n_3L'')/\Delta'_{23} \leqslant 0$	$(\Delta''_{23}K' - \Delta_{23}L'' + \Delta'_{23}R)/\Delta_{123} > 0$
I_{323}	$(-n_3K'' + p_3L')/\Delta'_{23} \leqslant 0$	$(-\Delta_{23}K'' + \Delta''_{23}L' + \Delta'_{23}R)/\Delta_{123} > 0$
I_{333}	$(m_3K'' - p_3L)/\Delta''_{23} \leqslant 0$	$(-\Delta_{23}K'' + \Delta'_{23}L + \Delta''_{23}R')/\Delta_{123} > 0$

Table 2. *Inequalities governing summation indices for series with leading factor* $\pm 1/\Delta_{123}$ *after reduction to common summand*

series	transformed two-term inequality	transformed three-term inequality
I_{124}	$(-n_3K + m_3L')/\Delta_{23} > 0$	$(\Delta'_{23}K + \Delta''_{23}L' - \Delta_{23}R'')/\Delta_{123} \leqslant 0$
I_{133}	$(-p_3K + m_3R'')/\Delta''_{23} > 0$	$(\Delta'_{23}K + \Delta''_{23}L' - \Delta_{23}R'')/\Delta_{123} > 0$
I_{224}	$(-n_3K + m_3L')/\Delta_{23} \leqslant 0$	$(\Delta'_{23}K + \Delta''_{23}L' - \Delta_{23}R'')/\Delta_{123} \leqslant 0$
I_{233}	$(p_3L' - n_3R'')/\Delta'_{23} \leqslant 0$	$(\Delta'_{23}K + \Delta''_{23}L' - \Delta_{23}R'')/\Delta_{123} > 0$
I_{323}	$(p_3L' - n_3R'')/\Delta'_{23} \leqslant 0$	$(\Delta'_{23}K + \Delta''_{23}L' - \Delta_{23}R'')/\Delta_{123} > 0$
I_{333}	$(-p_3K + m_3R'')/\Delta''_{23} \leqslant 0$	$(\Delta'_{23}K + \Delta''_{23}L' - \Delta_{23}R'')/\Delta_{123} > 0$

positive or not. For the six series under consideration here, these are summarized in table 1.

To effect a comparison of the series, we perform a variety of changes of summation indices, so that all summands have the common value $f_{124}(k,l,r)$ instead of some permutation of the order of these arguments. Thus, to the series I_{133}, we apply $r \to l$, $l \to r$; to I_{224}, apply $l \to k$, $k \to l$; to I_{233}, apply $r \to k$, $k \to l$, $l \to r$; to I_{323}, apply $r \to k$, $k \to r$; and to I_{333}, apply $l \to k$, $r \to l$ and $k \to r$. After these changes, each series $I_{124}, I_{133}, \ldots$ under consideration then has the common representation

$$\frac{\pm 1}{\Delta_{123}} \sum f_{124}(k,l,r),$$

only now the summation indices are governed by the appropriate entry in table 2.

Observe that in table 2, our suite of transformations has left only inequalities involving K, L' and R'', with those involving K', K'', L, etc., having been removed.

The series are now easily compared. We see that I_{133} and I_{333} (after change of summation indices) have complementary two-term inequalities, and identical three-term inequalities, so that I_{133} and I_{333} may be fused into a single series governed only by the inequality

$$(\Delta'_{23}K + \Delta''_{23}L' - \Delta_{23}R'')/\Delta_{123} > 0. \tag{4.66}$$

The inequalities governing I_{233} and I_{323} (after change of summation indices) are identical, and since I_{233} and I_{323} have leading factors of $1/\Delta_{123}$ and $-1/\Delta_{123}$, respectively, we see that these two series annihilate each other. The I_{124} and I_{224} series (after change of summation indices) have two-term inequalities that are complementary and identical three-term inequalities, so I_{124} and I_{224} may be fused into a single series governed only by the three-term inequality

$$(\Delta'_{23}K + \Delta''_{23}L' - \Delta_{23}R'')/\Delta_{123} \leqslant 0. \tag{4.67}$$

However, the inequalities governing the fused pairs of series, (4.66) and (4.67), are complementary so that these fused series may be collected together into a single series with no restrictions whatsoever, apart from the stipulation that the summation indices be non-negative integers. We have, therefore,

$$I_{\Delta_{123}} \equiv I_{124} + I_{133} + I_{224} + I_{233} + I_{323} + I_{333} = \frac{1}{\Delta_{123}} \sum_{k,l,r} f_{124}(k,l,r), \tag{4.68}$$

with f_{124} given in (4.25).

In a similar fashion, the series with leading factors of $1/\nu\Delta''_{12}$ sum together to produce a single unrestricted series, namely,

$$I_{\nu\Delta''_{12}} \equiv I_{113} + I_{132} + I_{332} = \frac{1}{\nu\Delta'_{12}} \sum_{k,l,r} f_{113}(k,l,r). \tag{4.69}$$

The two series with leading factor of $1/\eta\Delta_{23}$ fuse into the single unrestricted series

$$I_{\eta\Delta_{23}} \equiv I_{121} + I_{221} = \frac{1}{\eta\Delta_{23}} \sum_{k,l,r} f_{121}(k,l,r). \tag{4.70}$$

as do the series with leading factors of $\pm 1/\mu\Delta'_{13}$:

$$I_{\mu\Delta'_{13}} \equiv I_{213} + I_{231} + I_{313} + I_{321} = \frac{1}{\mu\Delta'_{13}} \sum_{k,l,r} f_{213}(k,l,r). \tag{4.71}$$

Finally, the series with leading factors of $\pm 1/\eta\Delta_{13}$ (I_{122}, I_{212} and I_{222}) mutually annihilate, as do the series with leading factors of $\pm 1/\mu\Delta'_{12}$ (I_{232} and I_{322}) and $\pm 1/\nu\Delta''_{13}$ (I_{131}, I_{312} and I_{331}).

Collecting our results, we arrive at the asymptotic expansion for $\lambda \to \infty$ (upon restoring the leading factor $\lambda^{-1/\mu-1/\nu-1/\eta}$)

$$I(\lambda) \sim \lambda^{-1/\mu-1/\nu-1/\eta}(I_{m_2\nu\eta} + I_{\mu n_3\eta} + I_{\mu\nu p_1} + I_{\mu\Delta'_{13}} + I_{\nu\Delta''_{12}} + I_{\eta\Delta_{23}} + I_{\Delta_{123}}). \tag{4.72}$$

The individual series in this compound expansion are to be found, in order of appearance, in (4.13), (4.40), (4.57), (4.71), (4.69), (4.70) and (4.68). The subscripts chosen for the component series indicate the corresponding face of the Newton diagram: $I_{m_2\nu\eta}$ is associated with the face with vertices P_2, $(0,\nu,0)$ and $(0,0,\eta)$; $I_{\mu\Delta'_{13}}$ is associated with the face with vertices $(\mu,0,0)$, P_1 and P_3; $I_{\Delta_{123}}$ is paired with the face with vertices P_1, P_2 and P_3, and so on.

After a linear-algebraic formulation of the results of this and the previous section, we address the problem of expansions that arise when faces of the Newton diagram fail to be triangular.

5. Linear-algebraic formulation

We record here a compact means by which the arguments of the Γ functions and associated quantities in the expansions of previous sections may be computed. This particular treatment applies only to the convex case where all faces of the

Newton diagram are triangular, and implicitly excludes the possibility of multiple poles and the logarithmic terms associated with the expansions that result from multiple poles. Our treatment parallels closely the general treatment described in (I, § 6 *b*). An important difference, however, is the fact that the faces cannot be so easily presented in the three-dimensional case; in particular, the ordering of faces that was exploited in (I, § 6 *b*) is no longer available to us.

To begin, let us assume we have $N > 1$ internal points, $P_1, P_2, \ldots, P_N$, each of which is a vertex and an extreme point of the Newton diagram. With N replacing k in (1.2) (k will be used exclusively as a summation index in this section), we see that there are $N + 3$ possible sequences of poles arising from the integrand of (1.2), one each from the Γ functions $\Gamma((1-\boldsymbol{m}\cdot\boldsymbol{t})/\mu)$, $\Gamma((1-\boldsymbol{n}\cdot\boldsymbol{t})/\nu)$ and $\Gamma((1-\boldsymbol{p}\cdot\boldsymbol{t})/\eta)$, and one from each of $\Gamma(t_1), \ldots, \Gamma(t_N)$. Each sequence arises from a (sometimes trivial) linear equation; in the order of the above listing of Γ functions, these are

$$\begin{array}{lcl} m_1t_1 + m_2t_2 + \cdots + m_Nt_N &=& 1 + \mu k, \\ n_1t_1 + n_2t_2 + \cdots + n_Nt_N &=& 1 + \nu l, \\ p_1t_1 + p_2t_2 + \cdots + p_Nt_N &=& 1 + \eta r, \\ t_1 &=& -s_1, \\ \quad t_2 &=& -s_2, \\ \quad\quad \ddots && \cdots \\ \quad\quad\quad t_N &=& -s_N, \end{array} \tag{5.1}$$

where $k, l, r, s_1, \ldots, s_N$ are non-negative integers. As displayed, this system is overdetermined.

Every face in the Newton diagram gives rise to an asymptotic series of the form (suppressing the leading factor of $\lambda^{-1/\mu-1/\nu-1/\eta}$)

$$\frac{1}{\Delta}\sum_{k,l,r,\ldots}\frac{(-1)^{k+l+r+\cdots}}{k!l!r!\cdots}\Gamma(\cdot)\Gamma(\cdot)\Gamma(\cdot)c_1^{-t_1}\cdots c_N^{-t_N}\lambda^{-\delta_1t_1-\cdots-\delta_Nt_N}, \tag{5.2}$$

with three Γ functions present in the summand. The form of the factor Δ depends on the particular vertices used to form the triangular face of the Newton diagram but is, in any event, the volume of the tetrahedral solid generated by the position vectors of the vertices defining the face.

To determine the arguments of the Γ functions in (5.2), we distinguish a number of cases: (i) two vertices of the face on the coordinate axes; (ii) one vertex on a coordinate axis; and (iii) no vertices concurrent with a coordinate axis. For convenience of presentation, by the term *omitted vertex*, let us mean a vertex on a coordinate axis which is not a vertex of the face, and by *omitted value*, we shall mean the number μ if $(\mu, 0, 0)$ is the omitted vertex, ν if $(0, \nu, 0)$ is the omitted vertex and η if $(0, 0, \eta)$ is the omitted vertex. The equation in (5.1) corresponding to an omitted value is, for μ, $m_1t_1 + \cdots + m_Nt_N = 1 + \mu k$, for ν, $n_1t_1 + \cdots + n_Nt_N = 1 + \nu l$, and so on.

In the first instance, (i), the face is formed from two of $(\mu, 0, 0)$, $(0, \nu, 0)$ and $(0, 0, \eta)$, and one of the internal points P_i, $1 \leqslant i \leqslant N$. If the omitted value is μ, then Δ in (5.2) is just $m_i\nu\eta$; if the omitted value is ν, then $\Delta = \mu n_i \eta$, and so on. The t-values to use in the Γ functions are computed from the linear system obtained from (5.1) by deleting the equation $t_i = -s_i$, and the equations corresponding to the non-omitted values. Thus, if $N = 3$, the internal point that is a vertex of the face is

P_i and μ is the omitted value, then the system to use in computing the t values is

$$\begin{aligned} m_1t_1 + m_2t_2 + m_3t_3 &= 1 + \mu k, \\ t_1 \qquad\qquad\qquad &= -s_1, \\ t_3 &= -s_3, \end{aligned}$$

and (5.2) would have $\Delta = m_2\nu\eta$ and summation indices k, s_1 and s_3. The Γ functions appearing in (5.2) are precisely those of the integrand of (1.2) whose poles are not to be found in the linear system above. For this particular example, we would use $\Gamma(t_2)\Gamma((1-\boldsymbol{n}\cdot\boldsymbol{t})/\nu)\Gamma((1-\boldsymbol{p}\cdot\boldsymbol{t})/\eta)$ evaluated at $t_1 = -s_1$, $t_3 = -s_3$ and the computed t_2 determined by our system.

For the second case, (ii), there are two omitted values, and two internal points, say P_i and P_j, serving as vertices of the face. The linear system to use in computing the t-values for the Γ functions is obtained from (5.1) by deleting $t_i = -s_i$, $t_j = -s_j$ and the equations corresponding to the non-omitted value. Thus, for $N = 3$, internal vertices P_1 and P_2 and (non-omitted) vertex $(0, \nu, 0)$, the relevant system to use is

$$\begin{aligned} m_1t_1 + m_2t_2 + m_3t_3 &= 1 + \mu k, \\ p_1t_1 + p_2t_2 + p_3t_3 &= 1 + \eta r, \\ t_3 &= -s_3. \end{aligned}$$

The factor Δ in this case would be the volume of the tetrahedral solid generated by the position vectors of the vertices P_1, P_2 and $(0, \nu, 0)$ (with suitable orientation, this is $\nu\Delta''_{12}$), the indices in the sum would be k, r and s_3, and the Γ functions appearing in (5.2) would be those from the integrand of (1.2) whose poles were not used in the above system. For our particular example, these are $\Gamma(t_1)\Gamma(t_2)\Gamma((1 - \boldsymbol{n}\cdot\boldsymbol{t})/\nu)$ evaluated at $t_3 = -s_3$ and the computed values of t_1 and t_2.

The final possibility, (iii), arises when all vertices are internal points, and all three of μ, ν and η are omitted values. If the internal points are P_i, P_j and P_k, then the appropriate linear system to use is obtained from (5.1) by deleting the equations $t_i = -s_i$, $t_j = -s_j$ and $t_k = -s_k$. The resulting system for $N = 3$ (with $\triangle P_1P_2P_3$ as the face) is

$$\begin{aligned} m_1t_1 + m_2t_2 + m_3t_3 &= 1 + \mu k, \\ n_1t_1 + n_2t_2 + n_3t_3 &= 1 + \nu l, \\ p_1t_1 + p_2t_2 + p_3t_3 &= 1 + \eta r. \end{aligned}$$

For this particular example, the factor $\Delta = \Delta_{123}$ (with appropriate orientation), the summation indices in (5.2) are k, l and r, and the Γ functions appearing in the sum are just $\Gamma(t_1)\Gamma(t_2)\Gamma(t_3)$ evaluated at the computed values of t_1, t_2 and t_3.

6. Non-triangular faces

One of the over-riding assumptions imposed on the previous sections is that the Newton diagram was formed only from triangular faces. In this section, we explore the consequences of the removal of this restriction, detailed for the setting of the convex case with two internal points. The analogue of this setting for two-dimensional Laplace integrals is that of adjacent collinear faces of the (two-dimensional) Newton diagram; cf. (I, § 5 *b*).

So let us assume that P_1 and P_2 are the internal points, and that the face joining $A \equiv (\mu, 0, 0)$, $B \equiv (0, 0, \eta)$, P_1 and P_2 is a single quadrilateral, as depicted in

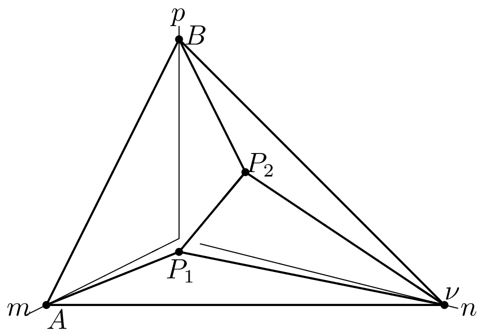

Figure 6. Newton diagram for two internal points, with a single quadrilateral face.

figure 6, and that the remaining three faces are triangles. The equation of the plane $\Pi((\mu,0,0),(0,0,\eta),P_1)$ is

$$\frac{m}{\mu}+\frac{(\mu\eta-m_1\eta-\mu p_1)}{\mu n_1\eta}n+\frac{p}{\eta}-1=0,$$

and if P_2 is to lie on this plane, then P_2 must therefore satisfy

$$-\frac{\Delta_{12}}{\mu}+\frac{\Delta'_{12}}{\eta}-N_{12}=0, \tag{6.1}$$

as can be seen upon applying the identities (A 1).

This restriction can be arrived at differently. Observe that the solid with vertices O, A, B, P_1 and P_2 (with quadrilateral face AP_1P_2B) has a volume that can be written as either

$$\text{volume}(O,A,P_1,P_2)+\text{volume}(O,A,P_2,B)\quad\text{or}$$
$$\text{volume}(O,B,P_1,P_2)+\text{volume}(O,B,A,P_1),$$

where $\text{volume}(Q_1,Q_2,Q_3,Q_4)$ is the volume of the tetrahedron with vertices Q_1,Q_2,Q_3,Q_4. Upon setting these two expressions equal, we arrive at (6.1).

For our integral (1.2) with $k=2$, we have δ_1 and δ_2 both positive, so if we displace the t_2 contour first to the right, we duplicate precisely the initial steps taken in § 3, encountering poles in the t_2 plane in sequences given by (3.1)–(3.2) and yielding asymptotic series I_1, I_2 and I_3 as displayed in (3.3)–(3.5). As was the case in § 3, we can produce asymptotic series I_{11}, I_{12} and I_{13}, given in (3.13), (3.14) and (3.15), respectively, and the asymptotic series I_{31}, I_{32} and I_{33}, as given in (3.24), (3.25) and (3.26), respectively. Some simplification of these results from § 3 is available to us, in view of (6.1), removing the k (or K) term from (3.17), and the corresponding term for I_{33}. Furthermore, as was the case in § 3, the two series I_{13} and I_{32} fuse together into a series which was denoted $I_{\nu\Delta''_{12}}$.

For I_2, however, the process changes. For the integrals in I_2, the logarithm of the modulus of the integrand is 0, measured against the asymptotic scale $\rho\log\rho$ (recall (3.6)), and the power of λ in the integrand is $-\boldsymbol{\delta}\cdot\boldsymbol{t}=-K'\delta_2/n_2$, i.e. λ no longer depends on the integration variable t_1 present in each of the integrals in I_2. Consequently, we may write

$$I_2=\frac{1}{\mu n_2\eta}\sum_k\frac{(-1)^k}{k!}\lambda^{-K'\delta_2/n_2}\cdot J_k, \tag{6.2}$$

where J_k is the Fox H-function (cf. Braaksma (1963, § 1.1)) given by

$$J_k \equiv \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \Gamma(t_1)\Gamma\left(\frac{K'-n_1t_1}{n_2}\right)\Gamma\left(\frac{n_2-m_2K'-\Delta_{12}t_1}{\mu n_2}\right) \times\Gamma\left(\frac{n_2-p_2K'+\Delta'_{12}t_1}{n_2\eta}\right)\mathrm{d}t_1, \tag{6.3}$$

to arrive at the asymptotic expansion

$$I(\lambda) \sim \lambda^{-1/\mu-1/\nu-1/\eta}(I_{11}+I_{12}+I_{31}+I_{33}+I_{\nu\Delta''_{12}}+I_2),$$

where, in order, the constituent series in the expansion are given by (3.13), (3.14), (3.24), (3.26), (3.15) (with no restrictions) and (6.2). This is, however, an unsatisfying result, and rather more can be said.

If we apply the order estimates (I, equation (2.1)), i.e. the well-known estimates for the logarithm of the Γ function of complex argument, to an integral of the form

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \Gamma(t)\Gamma(a-bt)\Gamma(c-dt)\Gamma(e+ft)\,\mathrm{d}t,$$

with a, b, c, d, e and f real, and b, d, f positive, then for $t=\rho e^{i\theta}$, ρ and θ real, the logarithm of the modulus of the integrand has the large-ρ behaviour

$$\rho\cos\theta\log\rho(1-b-d+f)+\rho\cos\theta\log\left(\frac{f^f}{b^bd^d}\right)+\rho(\theta\sin\theta+\cos\theta)(-1+b+d-f) \\ -\pi(b+d)\rho|\sin\theta|-\tfrac{1}{2}\log\rho+(a-\tfrac{1}{2})\log b\rho+(c-\tfrac{1}{2})\log d\rho+(e-\tfrac{1}{2})\log f\rho.$$

In the case of J_k, we have $a=K'/n_2$, $b=n_1/n_2$, $c=(n_2-m_2K')/\mu n_2$, $d=\Delta_{12}/\mu n_2$, $e=(n_2-p_2K')/n_2\eta$ and $f=\Delta'_{12}/n_2\eta$, so that with (6.1) holding, $1-b-d+f=0$. The preceding order estimate then reduces to the more compact

$$\rho\cos\theta\log\left(\frac{f^f}{b^bd^d}\right)+\rho(\theta\sin\theta+\cos\theta)-\pi(b+d)\rho|\sin\theta|+\mathcal{O}(\log\rho). \tag{6.4}$$

For sake of argument, let us assume the fraction in the logarithm is less than unity, so that this order estimate for $|\theta|<\frac{1}{2}\pi$ tends to $-\infty$ as $\rho\to\infty$. To obtain a convergent series representation, we displace the t_1 contour in (6.3) to the left, forming the series from residues of the integrand at

$$t_1^{(1)} = -l,$$
$$t_1^{(2)} = (-m_2K'+n_2L)/\Delta_{12} \leqslant 0, \tag{6.5}$$
$$t_1^{(3)} = (p_2K'-n_2L'')/\Delta'_{12} \leqslant 0. \tag{6.6}$$

As before, l is a non-negative integer and K', L and L'' are given in (2.7) and (3.12).

The series representation that we obtain for J_k is

$$J_k = \sigma_k - \frac{\mu n_2}{\Delta_{12}}\sigma'_k + \frac{n_2\eta}{\Delta'_{12}}\sigma''_k,$$

with

$$\sigma_k = \sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\Gamma\left(\frac{K'+n_1l}{n_2}\right)\Gamma\left(\frac{n_2-m_2K'+\Delta_{12}l}{\mu n_2}\right)\Gamma\left(\frac{n_2-p_2K'-\Delta'_{12}l}{n_2\eta}\right),$$

$$\sigma_k' = \sum_{l'} \frac{(-1)^l}{l!} \Gamma\left(\frac{-m_2K' + n_2L}{\Delta_{12}}\right) \Gamma\left(\frac{m_1K' - n_1L}{\Delta_{12}}\right) \Gamma\left(\frac{\Delta_{12} - \Delta_{12}''K' + \Delta_{12}'L}{\eta\Delta_{12}}\right),$$

$$\sigma_k'' = \sum_{l''} \frac{(-1)^l}{l!} \Gamma\left(\frac{p_2K' - n_2L''}{\Delta_{12}'}\right) \Gamma\left(\frac{-p_1K' + n_1L''}{\Delta_{12}'}\right) \Gamma\left(\frac{\Delta_{12}' - \Delta_{12}''K' + \Delta_{12}L''}{\mu\Delta_{12}'}\right),$$

where the prime on the index l in σ_k' indicates the restriction from (6.5), and the double prime on the index l in σ_k'' indicates the restriction stemming from (6.6). The convergence of the infinite series σ_k follows from theorem 1 of Braaksma (1963, § 6.1). The asymptotic series I_2 may therefore be written as

$$I_2 = \frac{1}{\mu n_2\eta} \sum_k \frac{(-1)^k}{k!} \sigma_k \lambda^{-\delta_2 K'/n_2} - \frac{1}{\eta\Delta_{12}} \sum_k \frac{(-1)^k}{k!} \sigma_k' \lambda^{-\delta_2 K'/n_2} + \frac{1}{\mu\Delta_{12}'} \sum_k \frac{(-1)^k}{k!} \sigma_k'' \lambda^{-\delta_2 K'/n_2}.$$

The term in I_2 associated with the finite series σ_k' cancels with the series I_{12} in view of the fact (easily verified with the aid of identities provided in the appendix)

$$\frac{1}{\Delta_{12}}\left(\frac{\Delta_{12}}{\nu} + \frac{\Delta_{12}''}{\eta} - M_{12}\right) = -\frac{\delta_2}{n_2},$$

and the term in I_2 associated with the finite series σ_k'' and the series I_{33} fuse into a single series with no restriction which we shall label $I_{\mu\Delta_{12}'}$ in view of the similar identity

$$\frac{1}{\Delta_{12}'}\left(\frac{\Delta_{12}''}{\mu} + \frac{\Delta_{12}'}{\nu} + P_{12}\right) = -\frac{\delta_2}{n_2}.$$

This fused series is defined by (3.26), and we may set the second term in (3.21) to zero in view of (6.1). As a result of these simplifications, the expansion of $I(\lambda)$ may be recast as

$$I(\lambda) \sim \lambda^{-1/\mu-1/\nu-1/\eta}\left(I_{m_2\nu\eta} + I_{\mu\Delta_{12}'} + I_{\nu\Delta_{12}''} + I_{\mu\nu p_1} + \frac{1}{\mu n_2\eta} \sum_k \frac{(-1)^k}{k!} \sigma_k \lambda^{-\delta_2 K'/n_2}\right),$$

as $\lambda \to \infty$.

Observe that the asymptotic scales of the series $I_{\mu\Delta_{12}'}$ and the series involving σ_k, respectively, associated with the triangular faces $\triangle AP_1P_2$ and $\triangle AP_2B$, are the same, and that these triangular faces constitute the quadrilateral face of the Newton diagram. Upon gathering together these two series, we arrive at the asymptotic expansion

$$I(\lambda) \sim \lambda^{-1/\mu-1/\nu-1/\eta}\left(I_{m_2\nu\eta} + I_{\nu\Delta_{12}''} + I_{\mu\nu p_1} + \sum_k \frac{(-1)^k}{k!}\left(\sum_l \frac{(-1)^l}{l!}\left[\frac{g_{\mu n_2\eta}(k,l)}{\mu n_2\eta} + \frac{g_{\mu\Delta_{12}'}(k,l)}{\mu\Delta_{12}'}\right]\right)\lambda^{-\delta_2 K'/n_2}\right),$$

for $\lambda \to \infty$, with

$$g_{\mu n_2\eta}(k,l) = \Gamma\left(\frac{K' + n_1 l}{n_2}\right) \Gamma\left(\frac{n_2 - m_2K' + \Delta_{12}l}{\mu n_2}\right) \Gamma\left(\frac{n_2 - p_2K' - \Delta_{12}'l}{n_2\eta}\right),$$

$$g_{\mu\Delta_{12}'}(k,l) = \Gamma\left(\frac{p_2K' - n_2L''}{\Delta_{12}'}\right) \Gamma\left(\frac{-p_1K' + n_1L''}{\Delta_{12}'}\right) \Gamma\left(\frac{\Delta_{12}' - \Delta_{12}''K' + \Delta_{12}L''}{\mu\Delta_{12}'}\right),$$

for non-negative integers k and l. The final form above for the asymptotic expansion of $I(\lambda)$ sees the term for the quadrilateral face represented as a weighted sum of the asymptotic series associated with each of the triangular faces that form the quadrilateral face. With this form, we have also preserved the notion of one component series per face of the Newton diagram.

If the fraction appearing in the logarithm in (6.4) were greater than unity, then instead of displacing the integration contour in (6.3) to the left, as we have done here, we would have shifted it to the right. Carrying through our formal series manipulations in this new set of circumstances, we would find a similar four-series compound expansion involving the same series for the triangular faces, but the quadrilateral face would have associated with it a series formed from 'triangular contributions' arising from splitting the quadrilateral along its other diagonal.

We can expect that other types of non-triangular faces in Newton diagrams would be associated with similar weighted sums of constituent triangular contributions, the precise nature of the decomposition being governed by order estimates of the moduli of integrands encountered through the application of our method.

7. Numerical examples

Example 7.1. We consider the integral with a single internal point given by

$$I(\lambda)=\int_0^\infty\int_0^\infty\int_0^\infty \exp[-\lambda(x^\mu+y^\nu+z^\eta+x^{m_1}y^{n_1}z^{p_1})]\,\mathrm{d}x\mathrm{d}y\mathrm{d}z, \tag{7.1}$$

where we assume that $\delta_1=1-m_1/\mu-n_1/\nu-p_1/\eta>0$, so that the vertex $P_1=(m_1,n_1,p_1)$ lies in front of the back face of the Newton diagram, and without loss of generality we have put the constant $c_1=1$. From (1.2), this has the Mellin representation

$$I(\lambda)=\frac{\lambda^{-1/\mu-1/\nu-1/\eta}}{\mu\nu\eta}\frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\Gamma(t)\Gamma\left(\frac{1-m_1t}{\mu}\right)\Gamma\left(\frac{1-n_1t}{\nu}\right)\Gamma\left(\frac{1-p_1t}{\eta}\right)\lambda^{-\delta_1 t}\,\mathrm{d}t, \tag{7.2}$$

which provides the analytic continuation of $I(\lambda)$ from the half-plane $\mathrm{Re}(\lambda)>0$ to a sector including the Fourier case with $\arg\lambda=\pm\frac{1}{2}\pi$. Although we are primarily concerned here with positive real λ, we remark that, like the integral (7.2), the asymptotic expansions developed for (7.1) are valid (in the Poincaré sense) for $\lambda\to\infty$ in the sector $|\arg\lambda|<\frac{1}{2}\pi(1+m_1/\mu+n_1/\nu+p_1/\eta)/\delta_1$; see (I, equation (3.7)).

When all the poles of the integrand in (7.2) are simple, the expansion of $I(\lambda)$ is given by (2.8). As an example of this simple-pole structure, we show in table 3 the results of numerical computations for the particular case $\mu=\nu=\eta=4$, with $m_1=\frac{2}{5}$, $n_1=\frac{1}{2}$ and $p_1=\frac{3}{5}$. The second column shows the value of $I(\lambda)$ determined by numerical evaluation of the integral (7.1) or from the representation of $I(\lambda)$ as a generalized hypergeometric function (cf. I, § 4 *b*). The third column shows the asymptotic value of $I(\lambda)$ obtained by optimally truncating each constituent asymptotic series in the expansion.

As mentioned in § 2, the incidence of higher order poles depends in part on the degree of symmetry of the Newton diagram. To illustrate this fact, let us first consider the most symmetrical situation with $\mu=\nu=\eta$ and $m_1=n_1=p_1$, so that the back face is an equilateral triangle and the vertex P_1 lies on the $(1,1,1)$ line. In this case

Table 3. *Comparison of the asymptotic values of $I(\lambda)$ with one internal point*

λ	$I(\lambda)$	asymptotic value
$\mu=\nu=\eta=4,\ m_1=\frac{2}{5},\ n_1=\frac{1}{2},\ p_1=\frac{3}{5},\ \delta_1=\frac{5}{8}$		
2.0×10^1	0.02234 75076	0.02155 91346
4.0×10^1	0.00876 25593	0.00876 06317
5.0×10^1	0.00639 39397	0.00639 38264
6.0×10^1	0.00492 03593	0.00492 02589
8.0×10^1	0.00322 97878	0.00322 97876
$\mu=\nu=\eta=4,\ m_1=1,\ n_1=1,\ p_1=1,\ \delta_1=\frac{1}{4}$		
1.0×10^2	0.01775 29863	0.01729 72404
2.0×10^2	0.01013 84495	0.01005 67565
5.0×10^2	0.00480 37523	0.00480 91387
1.0×10^3	0.00271 67652	0.00271 69815
5.0×10^3	0.00071 15152	0.00071 15151
$\mu=\nu=\eta=3,\ m_1=\frac{1}{2},\ n_1=\frac{1}{2},\ p_1=\frac{1}{2},\ \delta_1=\frac{1}{2}$		
5.0×10^1	0.00371 27773	0.00370 54488
8.0×10^1	0.00185 42982	0.00185 45509
1.0×10^2	0.00132 44252	0.00132 44556
1.5×10^2	0.00071 09237	0.00071 09234
2.0×10^2	0.00045 36285	0.00045 36285
$\mu=\nu=\eta=5,\ m_1=n_1=1,\ p_1=\frac{1}{2},\ \delta_1=\frac{1}{2}$		
5.0×10^1	0.03756 74263	0.03755 80382
8.0×10^1	0.02537 48359	0.02537 46738
1.0×10^2	0.02099 22663	0.02099 25949
1.5×10^2	0.01479 77289	0.01479 77269
2.0×10^2	0.01150 23481	0.01150 23481

the Mellin integral in (7.2) becomes

$$I(\lambda)=\frac{\lambda^{-3/\mu}}{\mu^3}\frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\Gamma(t)\Gamma^3\left(\frac{1-mt}{\mu}\right)\lambda^{-\delta t}\,\mathrm{d}t,\qquad \delta=1-3m/\mu,$$

where, for convenience, we have momentarily set $m_1=m$ and $\delta_1=\delta$. The three sequences of poles in the right half-plane in the general case have now collapsed into a single sequence of treble poles situated at $t=(1+\mu k)/m$, $k=0,1,2,\ldots$. The residues are given by the coefficient of x^{-1} in the Maclaurin expansion of

$$\Gamma\left(\frac{1+\mu k}{m}+x\right)\Gamma^3\left(-k-\frac{mx}{\mu}\right)\lambda^{-\delta x-\delta(1+\mu k)/m} = \pi^3(-1)^{k-1}\lambda^{-\delta(1+\mu k)/m}\cdot\frac{\lambda^{-\delta x}\Gamma(x+(1+\mu k)/m)}{\sin^3(\pi m x/\mu)\Gamma^3(k+1+mx/\mu)};$$

that is, by

$$(-1)^{k-1}\frac{(\mu/m)^3}{2(k!)^3}\Gamma\left(\frac{1+\mu k}{m}\right)\lambda^{-\delta(1+\mu k)/m}C_k(\lambda),$$

where

$$\begin{aligned}C_k(\lambda) = (\delta\log\lambda)^2 - 2\delta\log\lambda\left\{\psi\left(\frac{1+\mu k}{m}\right)-\frac{3m}{\mu}\psi(1+k)\right\}\\ +\psi'\left(\frac{1+\mu k}{m}\right)+\psi^2\left(\frac{1+\mu k}{m}\right)-\frac{6m}{\mu}\psi(1+k)\psi\left(\frac{1+\mu k}{m}\right)\\ +3\left(\frac{m}{\mu}\right)^2\{3\psi^2(1+k)-\psi'(1+k)\}+(\pi m/\mu)^2\end{aligned}$$

and ψ denotes the logarithmic derivative of the Γ function. When $\delta_1 > 0$, the expansion of $I(\lambda)$ in this case is therefore given by

$$I(\lambda) \sim \frac{\lambda^{-1/m}}{2m^3}\sum_{k=0}^{\infty}\frac{(-1)^k}{(k!)^3}\Gamma\left(\frac{1+\mu k}{m}\right)C_k(\lambda)\lambda^{-[(\mu/m)-3]k},$$

as $\lambda \to \infty$.

Examples of this highly symmetrical case with $\mu = 4$, $m = 1$ ($\delta_1 = \frac{1}{4}$) and $\mu = 3$, $m = \frac{1}{2}$ ($\delta_1 = \frac{1}{2}$) are also given in table 3. It will be observed that the two cases differ mainly in the proximity of the vertex P_1 to the back face, as can be seen from the corresponding values of δ_1 (cf. I, § 5 f). This results in the expansions being associated with different asymptotic scales, and hence different ranges of λ values.

The final example we give shows the type of expansion which arises when the $(1,1,1)$ line intersects an edge of the Newton diagram. The situation considered again has a symmetrical back face ($\mu = \nu = \eta$), but now with $m_1 = n_1 > p_1$, so that the edge connecting the vertex P_1 with $(0,0,\eta)$ meets the $(1,1,1)$ line. The integral (7.2) now possesses two sequences of poles in the right half-plane: a sequence of double poles at $t = (1+\mu k)/m$ and a sequence of simple poles at $t = (1+\mu k)/p$, $k = 0, 1, 2, \ldots$. Provided these two sequences have no point in common, the expansion of $I(\lambda)$ when $\delta_1 > 0$ is then given by

$$\begin{aligned}I(\lambda) \sim \frac{\lambda^{-3/\mu}}{\mu^2 p_1}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\Gamma\left(\frac{1+\mu k}{p_1}\right)\Gamma^2\left(\frac{p_1-m_1-m_1\mu k}{\mu p_1}\right)\lambda^{-\delta_1(1+\mu k)/p_1}\\ +\frac{\lambda^{-3/\mu}}{\mu m_1^2}\sum_{k=0}^{\infty}\frac{1}{(k!)^2}\Gamma\left(\frac{1+\mu k}{m}\right)\Gamma\left(\frac{m_1-p_1 k}{\mu m_1}\right)\lambda^{-\delta_1(1+\mu k)/m_1}\\ \times\left\{\delta_1\log\lambda-\psi\left(\frac{1+\mu k}{m_1}\right)+\frac{p_1}{\mu}\psi\left(\frac{m_1-p_1-p_1\mu k}{\mu m_1}\right)+\frac{2m_1}{\mu}\psi(1+k)\right\},\end{aligned} \tag{7.3}$$

as $\lambda \to \infty$. An example of the expansion (7.3) is shown in table 3 where $\mu = \nu = \eta = 5$

Table 4. *Comparison of the asymptotic values of $I(\lambda)$ with two internal points*

$\mu=3,\ \nu=4,\ \eta=5$
$(m_1,n_1,p_1)=(1.20,0.20,0.25)$
$(m_2,n_2,p_2)=(0.30,0.80,1.00)$
$\delta_1=\delta_2=\frac{1}{2}$

λ	$I(\lambda)$	asymptotic value
5.0×10^{1}	5.7415×10^{-3}	5.9135×10^{-3}
1.0×10^{2}	2.3182×10^{-3}	2.3170×10^{-3}
5.0×10^{2}	2.5801×10^{-4}	2.5828×10^{-4}
1.0×10^{3}	9.7595×10^{-5}	9.7652×10^{-5}
1.0×10^{4}	3.5915×10^{-6}	3.5997×10^{-6}

with $m_1=n_1=1$ and $p_1=\frac{1}{2}$ (so that $\delta_1=\frac{1}{2}$). This separation of the poles will certainly exist for integer values of m_1 and p_1, but not necessarily for non-integer values. In this case the expansion (7.3) would be modified by the formation of a sequence of treble poles which will generate additional terms involving $(\log\lambda)^2$ in the expansion.

As the symmetry of the Newton diagram is reduced the higher order poles become progressively more sparsely distributed. Even when the $(1,1,1)$ line punctures a face of the Newton diagram (cf. the first example in table 3 where all poles are simple), it is still possible to select values of the parameters which generate either double or treble poles, although the leading term of each expansion in these cases is always algebraic in λ.

Example 7.2. To illustrate a three-dimensional integral with two internal points, we consider

$$I(\lambda)=\int_0^\infty\int_0^\infty\int_0^\infty \exp[-\lambda(x^\mu+y^\nu+z^\eta+x^{m_1}y^{n_1}z^{p_1}+x^{m_2}y^{n_2}z^{p_2})]\,\mathrm{d}x\mathrm{d}y\mathrm{d}z,$$

where the parameter values will be assumed to correspond to a convex Newton diagram, with both internal points being counted as vertices, and the constants c_i have been chosen equal to unity. From (1.2), the Mellin representation is given by

$$I(\lambda)=\frac{\lambda^{-1/\mu-1/\nu-1/\eta}}{\mu\nu\eta}\left(\frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\right)^2\Gamma(t_1)\Gamma(t_2)\Gamma\left(\frac{1-m_1t_1-m_2t_2}{\mu}\right)\times\Gamma\left(\frac{1-n_1t_1-n_2t_2}{\nu}\right)\Gamma\left(\frac{1-p_1t_1-p_2t_2}{\eta}\right)\lambda^{-\delta_1t_1-\delta_2t_2}\,\mathrm{d}t_1\mathrm{d}t_2, \tag{7.4}$$

where $\delta_i=1-m_i/\mu-n_i/\nu-p_i/\eta$ $(i=1,2)$. This integral converges in the sector (see I, equation (3.7))

$$|\arg\lambda|<\min_{i=1,2}\{\tfrac{1}{2}\pi(1+m_i/\mu+n_i/\nu+p_i/\eta)/\delta_i\},$$

and so the asymptotic expansions developed in § 3 will hold (in the Poincaré sense) in this sector and, in particular, in the Fourier case $\arg\lambda=\pm\frac{1}{2}\pi$.

When all the poles of the integrand in (7.4) are simple, the expansion of $I(\lambda)$ is given by (3.27) and consists of five asymptotic series, one for each face of the

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

associated Newton diagram. Such a situation is given by $\mu = 3, \nu = 4$ and $\eta = 5$ with $P_1 = (1.20, 0.20, 0.25)$ and $P_2 = (0.30, 0.80, 1.00)$, for which it is easily verified that the convexity conditions in (3.8) and (3.9) are satisfied. The results of numerical computations for this case are presented in table 4.

8. Closing remarks

The summary of (I) applies to this paper, and so the remarks there will not be repeated here. We do add, however, that the interplay between the geometry of the Newton diagram of the phase of $I(\lambda)$ and the form of the compound expansions obtained in the present work is more involved than in (I), due in part to the richer structure of the diagram in three dimensions. This was especially apparent in the analysis of the case of three internal points (in § 4) where one face of the Newton diagram did not touch any coordinate axis.

The additional complexity of three-dimensional Newton diagrams can also be seen in the fact that there was no simple ansatz for dealing with non-triangular faces, in contrast to the method described in § 5. This analogue of the situation of collinear adjacent faces of the Newton diagram for double Laplace-type integrals (cf. (I, § 5 *b*)) requires the triangular decomposition of the non-triangular face and no simple geometric rule appeared to govern how the decomposition was to be undertaken. At present it appears the form of the analysis employed in § 6, with its attendent difficulties, is all that is available to treat this circumstance.

Of course, if retention of geometric information supplied by the Newton diagram is of no concern to the investigator, the method of representation as iterated Mellin–Barnes integrals can still be routinely applied to extract the algebraic asymptotic behaviour of $I(\lambda)$, even in higher-dimensional cases.

D.K. thanks the Division of Mathematical Sciences at the University of Abertay Dundee for its hospitality during the course of these investigations, and to the University of Lethbridge, its Faculty Association and NSERC Canada for financial support.

Appendix

We record here identities that proved useful in the course of some computations in the body of the paper.

For $i = 1, 2, 3$, define

$$\left.\begin{array}{lll} \Delta_{12} = m_1 n_2 - n_1 m_2, & \Delta_{13} = n_1 m_3 - m_1 n_3, & \Delta_{23} = n_2 m_3 - m_2 n_3, \\ \Delta'_{12} = n_1 p_2 - n_2 p_1, & \Delta'_{13} = n_1 p_3 - n_3 p_1, & \Delta'_{23} = n_2 p_3 - n_3 p_2, \\ \Delta''_{12} = m_1 p_2 - m_2 p_1, & \Delta''_{13} = m_1 p_3 - m_3 p_1, & \Delta''_{23} = m_3 p_2 - m_2 p_3, \end{array}\right\} \quad (\text{A}\,1)$$

$$\Delta_{123} = \begin{vmatrix} m_1 & n_1 & p_1 \\ m_2 & n_2 & p_2 \\ m_3 & n_3 & p_3 \end{vmatrix}, \quad (\text{A}\,2)$$

and for $i, j = 1, 2, 3$, define

$$M_{ij} = m_i - m_j, \quad N_{ij} = n_i - n_j, \quad P_{ij} = p_i - p_j. \quad (\text{A}\,3)$$

Then the following relationships hold.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Linear combinations of Δ_{ij}.

$$\left.\begin{aligned} m_1\Delta_{23} - m_2\Delta_{13} - m_3\Delta_{12} = 0, \\ n_1\Delta_{23} - n_2\Delta_{13} - n_3\Delta_{12} = 0, \\ p_1\Delta_{23} - p_2\Delta_{13} - p_3\Delta_{12} = \Delta_{123}. \end{aligned}\right\} \quad (A\,4)$$

Linear combinations of Δ'_{ij}.

$$\left.\begin{aligned} m_1\Delta'_{23} - m_2\Delta'_{13} + m_3\Delta'_{12} = \Delta_{123}, \\ n_1\Delta'_{23} - n_2\Delta'_{13} + n_3\Delta'_{12} = 0, \\ p_1\Delta'_{23} - p_2\Delta'_{13} + p_3\Delta'_{12} = 0. \end{aligned}\right\} \quad (A\,5)$$

Linear combinations of Δ''_{ij}.

$$\left.\begin{aligned} m_1\Delta''_{23} + m_2\Delta''_{13} - m_3\Delta''_{12} = 0, \\ n_1\Delta''_{23} + n_2\Delta''_{13} - n_3\Delta''_{12} = \Delta_{123}, \\ p_1\Delta''_{23} + p_2\Delta''_{13} - p_3\Delta''_{12} = 0. \end{aligned}\right\} \quad (A\,6)$$

Mixed linear combinations.

$$\left.\begin{aligned} m_3\Delta'_{12} - n_3\Delta''_{12} + p_3\Delta_{12} = \Delta_{123}, \\ m_3\Delta'_{23} + n_3\Delta''_{23} - p_3\Delta_{23} = 0, \\ m_3\Delta'_{13} - n_3\Delta''_{13} - p_3\Delta_{13} = 0, \\ m_2\Delta'_{12} - n_2\Delta''_{12} + p_2\Delta_{12} = 0 \\ m_2\Delta'_{13} - n_2\Delta''_{13} - p_2\Delta_{13} = -\Delta_{123}. \end{aligned}\right\} \quad (A\,7)$$

Linear combinations of δ_i and m_j.

$$\left.\begin{aligned} m_2\delta_1 - m_1\delta_2 = \Delta_{12}/\nu + \Delta''_{12}/\eta - M_{12}, \\ m_3\delta_1 - m_1\delta_3 = -\Delta_{13}/\nu + \Delta''_{13}/\eta - M_{13}, \\ m_3\delta_2 - m_2\delta_3 = -\Delta_{23}/\nu - \Delta''_{23}/\eta - M_{23}. \end{aligned}\right\} \quad (A\,8)$$

Linear combinations of δ_i and n_j.

$$\left.\begin{aligned} n_2\delta_1 - n_1\delta_2 = -\Delta_{12}/\mu + \Delta'_{12}/\eta - N_{12}, \\ n_3\delta_1 - n_1\delta_3 = \Delta_{13}/\mu + \Delta'_{13}/\eta - N_{13}, \\ n_3\delta_2 - n_2\delta_3 = \Delta_{23}/\mu + \Delta''_{23}/\eta - N_{23}. \end{aligned}\right\} \quad (A\,9)$$

Linear combinations of δ_i and p_j.

$$\left.\begin{aligned} p_2\delta_1 - p_1\delta_2 = -\Delta''_{12}/\mu - \Delta'_{12}/\nu - P_{12}, \\ p_3\delta_1 - p_1\delta_3 = -\Delta''_{13}/\mu - \Delta'_{13}/\nu - P_{13}, \\ p_3\delta_2 - p_2\delta_3 = \Delta''_{23}/\mu - \Delta'_{23}/\nu - P_{23}. \end{aligned}\right\} \quad (A\,10)$$

Products of Δ_{ij}, Δ'_{ij} and Δ''_{ij}.

$$\left.\begin{aligned} \Delta''_{13}\Delta_{23} + \Delta''_{23}\Delta_{13} = m_3\Delta_{123}, \\ \Delta'_{23}\Delta''_{13} + \Delta''_{23}\Delta'_{13} = p_3\Delta_{123}, \\ \Delta'_{23}\Delta''_{12} + \Delta'_{12}\Delta''_{23} = p_2\Delta_{123}, \\ \Delta_{13}\Delta'_{23} - \Delta_{23}\Delta'_{13} = -n_3\Delta_{123}, \\ \Delta_{12}\Delta'_{23} + \Delta_{23}\Delta'_{12} = n_2\Delta_{123}, \\ \Delta_{12}\Delta''_{23} - \Delta_{23}\Delta''_{12} = -m_2\Delta_{123}. \end{aligned}\right\} \quad (A\,11)$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Products of δ_i and Δ_{jk}.

$$\left.\begin{aligned}-\Delta_{23}\delta_1+\Delta_{13}\delta_2+\Delta_{12}\delta_3&=-\Delta_{123}/\eta+\Delta_{12}+\Delta_{13}-\Delta_{23},\\-\Delta'_{23}\delta_1+\Delta'_{13}\delta_2-\Delta'_{12}\delta_3&=\Delta_{123}/\mu-\Delta'_{12}+\Delta'_{13}-\Delta'_{23},\\-\Delta''_{23}\delta_1-\Delta''_{13}\delta_2+\Delta''_{12}\delta_3&=\Delta_{123}/\nu+\Delta''_{12}-\Delta''_{13}-\Delta''_{23}.\end{aligned}\right\}\quad\text{(A 12)}$$

References

Braaksma, B. L. J. 1963 Asymptotic expansions and analytic continuations for a class of Barnes integrals. *Compositio Math.* **15**, 239–341.

Kaminski, D. & Paris, R. B. 1998 Asymptotics of a class of multidimensional Laplace-type integrals. I. Double integrals. *Phil. Trans. R. Soc. Lond.* A **356**, 583–623. (Preceding paper.)

MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES
THE ROYAL SOCIETY A
PHILOSOPHICAL TRANSACTIONS OF